On necessary and sufficient conditions of the BV quantization of a generic Lagrangian field system

Denis Bashkirov[1], **Giovanni Giachetta**[2], **Luigi Mangiarotti**[2], **Gennadi Sardanashvily**[1]

[1] Department of Theoretical Physics, Physics Faculty, Moscow State University, 117234 Moscow, Russia

[2] Department of Mathematics and Informatics, University of Camerino, 62032 Camerino (MC), Italy

Abstract: We address the problem of extending an original field Lagrangian to ghosts and antifields in order to satisfy the master equation in the framework of the BV quantization of Lagrangian field systems. This extension essentially depends on the degeneracy of an original Lagrangian whose Euler–Lagrange operator generally obeys the Noether identities which need not be independent, but satisfy the first-stage Noether identities, and so on. A generic Lagrangian system of even and odd fields on an arbitrary smooth manifold is examined in the algebraic terms of the Grassmann-graded variational bicomplex. We state the necessary and sufficient condition for the existence of the exact antifield Koszul–Tate complex whose boundary operator provides all the Noether and higher-stage Noether identities of an original Lagrangian system. The Noether inverse second theorem that we prove associates to this Koszul–Tate complex the sequence of ghosts whose ascent operator provides the gauge and higher-stage gauge supersymmetries of an original Lagrangian. We show that an original Lagrangian is extended to a solution of the master equation if this ascent operator admits a nilpotent extension and only if it is extended to an operator nilpotent on the shell.

1 Introduction

As is well known, the Batalin-Vilkovisky (henceforth BV) quantization of a Lagrangian field system essentially depends on the analysis of its degeneracy [2, 8, 16, 20]. A Lagrangian system is said to be degenerate if its Euler–Lagrange operator obeys non-trivial Noether identities. They need not be independent, but satisfy the first-stage Noether identities, which in turn are subject to the second-stage ones, and so on. The hierarchy of reducible Noether identities characterizes the degeneracy of a Lagrangian system in full. The Noether second theorem states the relation between the Noether identities and the gauge symmetries of a Lagrangian system. If Noether identities and gauge symmetries are finitely generated, they are parameterized by the modules of antifields and ghosts, respectively. An original Lagrangian is extended to these antifields and ghosts in order to satisfy the so-called classical master equation. This extended Lagrangian is the main ingredient in the BV quantization procedure.

It should be noted that the notion of reducible Noether identities has come from that of reducible constraints. Their Koszul–Tate complex has been invented by analogy with that of constraints [12] under a rather restrictive regularity condition that field equations as well as Noether identities of arbitrary stage can be locally separated into the independent and

dependent ones [2, 12]. This condition has also come from the case of constraints locally given by a finite number of functions which the inverse mapping theorem is applied to. A problem is that, in contrast with constraints, Noether and higher-stage Noether identities are differential operators. They are locally given by a set of functions and their jet prolongations on an infinite order jet manifold. Since the latter is a Fréchet, but not Banach manifold, the inverse mapping theorem fails to be valid.

Following the general notion of Noether identities of differential operators [6, 24], we here address a generic degenerate Lagrangian system of even and odd fields on an arbitrary smooth manifold. Dealing with odd fields, we follow the algebraic description of Lagrangian systems in terms of the Grassmann-graded variational bicomplex [2, 5, 18]. Theorem 2.2 provides its relevant cohomology.

We show that, if Noether and higher-stage Noether identities are assumed to be finitely generated and iff a certain homology regularity condition (Definition 3.6) holds, one can associate to the Euler–Lagrange operator of a degenerate Grassmann-graded Lagrangian system the exact Koszul–Tate complex (3.33) of antifields whose boundary operator (3.34) provides all the Noether and higher-stage Noether identities of an original Lagrangian system (Theorem 3.7).

The Noether second theorem relates the Noether and higher-stage Noether identities to the gauge and higher-stage gauge symmetries and supersymmetries of a Lagrangian system [4, 5, 15]. We prove its variant (Theorem 4.1) which associates to the above mentioned Koszul–Tate complex the sequence (4.4), graded in ghosts, whose ascent operator (4.5) provides gauge and higher-stage gauge supersymmetries of an original Lagrangian system. We agree to call it the total gauge operator, acting both on original fields and ghosts. This operator need not be nilpotent. If it admits a nilpotent (resp. nilpotent on the shell) extension, one can say that gauge and higher-stage gauge supersymmetries of an original Lagrangian system constitute an algebra (resp. an algebra on the shell).

Extending an original Lagrangian system to the above mentioned ghosts and antifields, we come to a Lagrangian system whose Lagrangian can satisfy the particular condition called the classical master equation (Proposition 5.1). We show that an original Lagrangian is extended to a nontrivial solution of the master equation only if the above mentioned total gauge operator (4.5) is extended to an operator nilpotent on the shell (Theorem 5.2) and if this operator admits a nilpotent extension (Theorem 5.3).

Note that the proof of Theorem 5.3 appeals to the above mentioned homology regularity condition, and states something more. Given a nilpotent extension (4.16) of the total gauge operator, a desired solution of the master equation is obtained at once by the formula (5.22) (or (5.23)). It is affine in antifields, and this fact is essential for the further gauge fixing procedure. We thus may conclude that the study of a classical Lagrangian field system for the purpose of its BV quantization mainly reduces to constructing the total gauge operator (4.5) and its nilpotent extension.

In Section 6, an example coming from topological BF theory is examined in detail. It is a reducible degenerate Lagrangian system whose total gauge operator is nilpotent and, thus, provides an extension of the original Lagrangian (6.1) of the topological BF theory to the solution (6.10) of the master equation.

2 Grassmann-graded Lagrangian systems

We describe Lagrangian systems of even and odd variables in algebraic terms of the Grassmann-graded variational bicomplex [2, 5, 18], generalizing the well-known variational bicomplex for even Lagrangian systems on fiber bundles [1, 17, 25].

Remark 2.1. Smooth manifolds throughout are real, finite-dimensional, Hausdorff, second-countable (hence, paracompact) and connected. Graded manifolds with structure sheaves of Grassmann algebras of finite rank are only considered. By Λ, Σ, Ξ, are denoted symmetric multi-indices, e.g., $\Lambda=(\lambda_1...\lambda_k)$, $\lambda+\Lambda=(\lambda\lambda_1...\lambda_k)$. By a summation over a multi-index $\Lambda=(\lambda_1...\lambda_k)$ is meant separate summation over each index λ_i.

Let $Y\to X$, $\dim X=n$, be a fiber bundle. The jet manifolds J^rY of its sections form the inverse system

$$X\overset{\pi}{\longleftarrow} Y\overset{\pi^1_0}{\longleftarrow} J^1Y\longleftarrow\cdots J^{r-1}Y\overset{\pi^r_{r-1}}{\longleftarrow} J^rY\longleftarrow\cdots, \tag{2.1}$$

where π^r_{r-1} are affine bundles. Its projective limit $(J^\infty Y;\pi^\infty_r: J^\infty Y\to J^rY)$ is a paracompact Fréchet manifold. A bundle atlas $\{(U_Y;x^\lambda,y^i)\}$ of $Y\to X$ induces the coordinate atlas

$$\begin{aligned}
&\{((\pi^\infty_0)^{-1}(U_Y);x^\lambda,y^i_\Lambda)\}, \qquad {y'}^i_{\lambda+\Lambda}=\frac{\partial x^\mu}{\partial x'^\lambda}d_\mu y'^i_\Lambda, \qquad 0\leq|\Lambda|, \\
&d_\lambda=\partial_\lambda+\sum_{0\leq|\Lambda|} y^i_{\lambda+\Lambda}\partial^\Lambda_i, \qquad d_\Lambda=d_{\lambda_1}\circ\cdots\circ d_{\lambda_k},
\end{aligned}\tag{2.2}$$

of $J^\infty Y$, where d_λ are total derivatives. The inverse system (2.1) yields the direct system

$$\mathcal{O}^*X\overset{\pi^*}{\longrightarrow}\mathcal{O}^*Y\overset{\pi^{1*}_0}{\longrightarrow}\mathcal{O}^*_1Y\longrightarrow\cdots\mathcal{O}^*_{r-1}Y\overset{{\pi^r_{r-1}}^*}{\longrightarrow}\mathcal{O}^*_rY\longrightarrow\cdots \tag{2.3}$$

of graded differential algebras (henceforth GDAs) $\mathcal{O}^*_rY$ of exterior forms on X, Y and jet manifolds J^rY with respect to the pull-back monomorphisms ${\pi^r_{r-1}}^*$. Its direct limit is the GDA $\mathcal{O}^*_\infty Y$ of all exterior forms on finite order jet manifolds modulo the pull-back identification. The GDA $\mathcal{O}^*_\infty Y$ is split into the above mentioned variational bicomplex describing Lagrangian systems of even fields on a fiber bundle $Y\to X$.

Treating odd fields, we appeal to forthcoming Theorem 2.1, which is a corollary of the Batchelor theorem [3] and the Serre-Swan theorem for an arbitrary smooth manifold [19, 23].

Theorem 2.1. *A Grassmann algebra $\mathcal{A}$ over the ring $C^\infty(Z)$ of smooth real functions on a manifold Z is isomorphic to the algebra of graded functions on a graded manifold with a body Z iff it is the exterior algebra of some projective $C^\infty(Z)$-module of finite rank [6].*

Recall that the above mentioned Batchelor theorem states an isomorphism of a graded manifold $(Z,\mathfrak{A})$ with a body Z to the particular one $(Z,\mathfrak{A}_Q)$ with the structure sheaf $\mathfrak{A}_Q$ of germs of sections of the exterior bundle

$$\wedge Q^*=\mathbb{R}\underset{Z}{\oplus}Q^*\underset{Z}{\oplus}\overset{2}{\wedge}Q^*\underset{Z}{\oplus}\cdots,$$

where Q^* is the dual of some vector bundle $Q\to Z$. In field models, Batchelor's isomorphism is usually fixed from the beginning. Let us call $(Z,\mathfrak{A}_Q)$ the simple graded manifold modelled

over Q. Its ring $\mathcal{A}_Q$ of graded functions consists of sections of $\wedge Q^*$. The following bigraded differential algebra (henceforth BGDA) $\mathcal{S}^*[Q;Z]$ is associated to $(Z,\mathfrak{A}_Q)$ [3, 19].

Let $\mathfrak{d}\mathfrak{A}_Q$ be the sheaf of graded derivations of $\mathfrak{A}_Q$. Its global sections make up the real Lie superalgebra $\mathfrak{d}\mathcal{A}_Q$ of (left) graded derivations of the $\mathbb{R}$-ring $\mathcal{A}_Q$, i.e.,

$$u(ff')=u(f)f'+(-1)^{[u][f]}fu(f'), \qquad f,f'\in\mathcal{A}_Q, \qquad u\in\mathfrak{d}\mathcal{A}_Q,$$

where the symbol [.] stands for the Grassmann parity. Then the Chevalley–Eilenberg complex of $\mathfrak{d}\mathcal{A}_Q$ with coefficients in $\mathcal{A}_Q$ can be constructed [13]. Its subcomplex $\mathcal{S}^*[Q;Z]$ of $\mathcal{A}_Q$-linear morphisms is the Grassmann-graded Chevalley–Eilenberg differential calculus

$$0\to\mathbb{R}\to\mathcal{A}_Q\xrightarrow{d}\mathcal{S}^1[Q;Z]\xrightarrow{d}\cdots\mathcal{S}^k[Q;Z]\xrightarrow{d}\cdots$$

over a $\mathbb{Z}_2$-graded commutative $\mathbb{R}$-ring $\mathcal{A}_Q$ [19]. The graded exterior product $\wedge$ and the Chevalley–Eilenberg coboundary operator d make $\mathcal{S}^*[Q;Z]$ into a BGDA

$$\phi\wedge\phi'=(-1)^{|\phi||\phi'|+[\phi][\phi']}\phi'\wedge\phi, \qquad d(\phi\wedge\phi')=d\phi\wedge\phi'+(-1)^{|\phi|}\phi\wedge d\phi', \tag{2.4}$$

where $|.|$ denotes the form degree. Note that $\mathcal{S}^*[Q;Z]$ is a minimal differential calculus over $\mathcal{A}_Q$, i.e., it is generated by elements df, $f\in\mathcal{A}_Q$. There is the natural monomorphism $\mathcal{O}^*Z\to\mathcal{S}^*[Q;Z]$.

One can think of elements of the BGDA $\mathcal{S}^*[Q;Z]$ as being graded exterior forms on a manifold Z as follows. Given an open subset $U\subset Z$, let $\mathcal{A}_U$ be the Grassmann algebra of sections of the sheaf $\mathfrak{A}_Q$ over U, and let $\mathcal{S}^*[Q;U]$ be the Chevalley–Eilenberg differential calculus over $\mathcal{A}_U$. With another open subset $U'\subset U$, the restriction morphism $\mathcal{A}_U\to\mathcal{A}_{U'}$ yields a homomorphism of BGDAs $\mathcal{S}^*[Q;U]\to\mathcal{S}^*[Q;U']$. Thus, we obtain the presheaf $\{U,\mathcal{S}^*[Q;U]\}$ of BGDAs on a manifold Z and the sheaf $\mathfrak{S}^*[Q;Z]$ of BGDAs of germs of this presheaf. Since $\{U,\mathcal{A}_U\}$ is the canonical presheaf of $\mathfrak{A}_Q$, the canonical presheaf of $\mathfrak{S}^*[Q;Z]$ is $\{U,\mathcal{S}^*[Q;U]\}$. In particular, $\mathcal{S}^*[Q;Z]$ is the BGDA of global sections of the sheaf $\mathfrak{S}^*[Q;Z]$, and there is the restriction morphism $\mathcal{S}^*[Q;Z]\to\mathcal{S}^*[Q;U]$ for any open $U\subset Z$.

Due to this restriction morphism, elements of $\mathcal{S}^*[Q;Z]$ can be written in the following local form. Given bundle coordinates (z^A,q^a) on Q and the corresponding fiber basis $\{c^a\}$ for $Q^*\to X$, the tuple (z^A,c^a) is called a local basis for the graded manifold $(Z,\mathfrak{A}_Q)$. With respect to this basis, graded functions read

$$f=\sum_{0\le k}\frac{1}{k!}f_{a_1\ldots a_k}c^{a_1}\cdots c^{a_k}. \qquad f_{a_1\ldots a_k}\in C^\infty(Z). \tag{2.5}$$

Due to the canonical splitting $VQ=Q\times Q$, the fiber basis $\{\partial_a\}$ for vertical tangent bundle $VQ\to Q$ of $Q\to Z$ is the dual of $\{c^a\}$. Then graded derivations take the local form $u=u^A\partial_A+u^a\partial_a$, where u^A,u^a are local graded functions. They act on graded functions (2.5) by the rule

$$u(f_{a\ldots b}c^a\cdots c^b)=u^A\partial_A(f_{a\ldots b})c^a\cdots c^b+u^df_{a\ldots b}\partial_d\rfloor(c^a\cdots c^b).$$

Relative to the dual bases $\{dz^A\}$ for T^*Z and $\{dc^b\}$ for Q^*, graded one-forms read $\phi=\phi_Adz^A+\phi_adc^a$. The duality morphism and the graded exterior differential d take the form

$$u\rfloor\phi=u^A\phi_A+(-1)^{[\phi_a]}u^a\phi_a, \qquad d\phi=dz^A\wedge\partial_A\phi+dc^a\wedge\partial_a\phi.$$

Remark 2.2. One also deals with right graded derivations $\overleftarrow{u}$ of graded functions and the right graded exterior differential $\overleftarrow{d}$. They read

$$\overleftarrow{u}(ff') = f\,\overleftarrow{u}(f') + (-1)^{[\overleftarrow{u}][f']}\,\overleftarrow{u}(f)f', \qquad f, f' \in \mathcal{A}_Q,$$
$$\overleftarrow{u}(f) = \partial_A(f)u^A + \overleftarrow{\partial}_d(f)u^d, \qquad \overleftarrow{\partial}_d(f_{a\ldots b}c^a\cdots c^b) = f_{a\ldots b}(c^a\cdots c^b)\lfloor\overleftarrow{\partial}_d,$$
$$\overleftarrow{d}(\phi) = \partial_A(\phi)\wedge dz^A + \overleftarrow{\partial}_a(\phi)\wedge dc^a, \qquad \phi \in \mathcal{S}^*[Q;Z].$$

If $Y \to X$ is an affine bundle, the total algebra of even and odd fields has been defined as the product of the polynomial subalgebra of the GDA $\mathcal{O}^*_\infty Y$ and some BGDA of graded exterior forms on graded manifolds whose body is X [5, 18]. Since $Y \to X$ here need not be affine, we consider graded manifolds whose bodies are Y and jet manifolds J^rY [6]. This definition of jets of odd fields differs from that of jets of a graded commutative ring [19] and jets of a graded fiber bundle [21], but reproduces the heuristic notion of jets of odd ghosts in the Lagrangian BRST theory [2, 10].

Given a vector bundle $F \to X$, let us consider the simple graded manifold $(J^rY, \mathfrak{A}_{F_r})$ modelled over the pull-back $F_r = J^rY \times_X J^rF$ onto J^rY of the jet bundle $J^rF \to X$. There is an epimorphism of graded manifolds

$$(J^{r+1}Y, \mathfrak{A}_{F_{r+1}}) \to (J^rY, \mathfrak{A}_{F_r}),$$

regarded as local-ringed spaces. It consists of the surjection π^{r+1}_r and the sheaf monomorphism $\pi^{r+1*}_r\mathfrak{A}_{F_r} \to \mathfrak{A}_{F_{r+1}}$, where $\pi^{r+1*}_r\mathfrak{A}_{F_r}$ is the pull-back onto $J^{r+1}Y$ of the topological fiber bundle $\mathfrak{A}_{F_r} \to J^rY$. This sheaf monomorphism induces the monomorphism of the canonical presheaves $\overline{\mathfrak{A}}_{F_r} \to \overline{\mathfrak{A}}_{F_{r+1}}$, which associates to each open subset $U \subset J^{r+1}Y$ the ring of sections of $\mathfrak{A}_{F_r}$ over $\pi^{r+1}_r(U)$. Accordingly, there is the monomorphism of graded commutative rings $\mathcal{A}_{F_r} \to \mathcal{A}_{F_{r+1}}$. Then this monomorphism yields the monomorphism of BGDAs

$$\mathcal{S}^*[F_r; J^rY] \to \mathcal{S}^*[F_{r+1}; J^{r+1}Y]. \tag{2.6}$$

As a consequence, we have the direct system of BGDAs

$$\mathcal{S}^*[Y \underset{X}{\times} F; Y] \longrightarrow \mathcal{S}^*[F_1; J^1Y] \longrightarrow \cdots \mathcal{S}^*[F_r; J^rY] \longrightarrow \cdots, \tag{2.7}$$

whose direct limit $\mathcal{S}^*_\infty[F;Y]$ is a BGDA of all graded differential forms on jet manifolds J^rY modulo monomorphisms (2.6). Its elements obey the relations (2.4). The monomorphisms $\mathcal{O}^*_rY \to \mathcal{S}^*[F_r; J^rY]$ provide a monomorphism of the direct system (2.3) to the direct system (2.7) and, consequently, the monomorphism $\mathcal{O}^*_\infty Y \to \mathcal{S}^*_\infty[F;Y]$ of their direct limits. In particular, $\mathcal{S}^*_\infty[F;Y]$ is an $\mathcal{O}^0_\infty Y$-algebra.

It is the BGDA $\mathcal{S}^*_\infty[F;Y]$ which provides algebraic description of Lagrangian systems of even and odd fields. If $Y \to X$ is an affine bundle, we recover the BGDA introduced in [5, 18] by restricting the ring $\mathcal{O}^0_\infty Y$ to its subring $\mathcal{P}^0_\infty Y$ of polynomial functions, but now elements of $\mathcal{S}^*_\infty[F;Y]$ are graded exterior forms on $J^\infty Y$. Indeed, let $\mathfrak{S}^*[F_r; J^rY]$ be the sheaf of BGDAs on J^rY and $\overline{\mathfrak{S}}^*[F_r; J^rY]$ its canonical presheaf whose elements are the Chevalley–Eilenberg

differential calculus over elements of the presheaf $\overline{\mathfrak{A}}_{F_r}$. Then the presheaf monomorphisms $\overline{\mathfrak{A}}_{F_r} \to \overline{\mathfrak{A}}_{F_{r+1}}$ yield the direct system of presheaves

$$\overline{\mathfrak{S}}^*[Y\times F;Y] \longrightarrow \overline{\mathfrak{S}}^*[F_1;J^1Y] \longrightarrow \cdots \overline{\mathfrak{S}}^*[F_r;J^rY] \longrightarrow \cdots, \tag{2.8}$$

whose direct limit $\overline{\mathfrak{S}}^*_\infty[F;Y]$ is a presheaf of BGDAs on the infinite order jet manifold $J^\infty Y$. Let $\mathfrak{T}^*_\infty[F;Y]$ be the sheaf of BGDAs of germs of the presheaf $\overline{\mathfrak{S}}^*_\infty[F;Y]$. The structure module $\Gamma(\mathfrak{T}^*_\infty[F;Y])$ of sections of $\mathfrak{T}^*_\infty[F;Y]$ is a BGDA such that, given an element $\phi\in\Gamma(\mathfrak{T}^*_\infty[F;Y])$ and a point $z\in J^\infty Y$, there exist an open neighbourhood U of z and a graded exterior form $\phi^{(k)}$ on some finite order jet manifold J^kY so that $\phi|_U=\pi^{\infty *}_k\phi^{(k)}|_U$. In particular, there is the monomorphism $\mathcal{S}^*_\infty[F;Y]\to\Gamma(\mathfrak{T}^*_\infty[F;Y])$.

Due to this monomorphism, one can restrict $\mathcal{S}^*_\infty[F;Y]$ to the coordinate chart (2.2) and say that $\mathcal{S}^*_\infty[F;Y]$ as an $\mathcal{O}^0_\infty Y$-algebra is locally generated by the elements

$$(1, c^a_\Lambda, dx^\lambda, \theta^a_\Lambda = dc^a_\Lambda - c^a_{\lambda+\Lambda}dx^\lambda, \theta^i_\Lambda = dy^i_\Lambda - y^i_{\lambda+\Lambda}dx^\lambda), \qquad 0\leq|\Lambda|.$$

One call (y^i,c^a) the local basis for $\mathcal{S}^*_\infty[F;Y]$. We further use the collective symbol s^A for its elements, together with the notation s^A_Λ, $\theta^A_\Lambda=ds^A_\Lambda - s^A_{\lambda+\Lambda}dx^\lambda$, and $[A]=[s^A]$.

Remark 2.3. Given local graded functions f^Λ and a graded form Φ, there are useful relations

$$\sum_{0\leq|\Lambda|\leq k}(-1)^{|\Lambda|}d_\Lambda(f^\Lambda\Phi)=\sum_{0\leq|\Lambda|\leq k}\eta(f)^\Lambda d_\Lambda\Phi, \tag{2.9}$$

$$\eta(f)^\Lambda = \sum_{0\leq|\Sigma|\leq k-|\Lambda|}(-1)^{|\Sigma+\Lambda|}C^{|\Sigma|}_{|\Sigma+\Lambda|}d_\Sigma f^{\Sigma+\Lambda}, \qquad C^a_b=\frac{b!}{a!(b-a)!}, \tag{2.10}$$

$$(\eta\circ\eta)(f)^\Lambda = f^\Lambda. \tag{2.11}$$

The BGDA $\mathcal{S}^*_\infty[F;Y]$ is decomposed into $\mathcal{S}^0_\infty[F;Y]$-modules $\mathcal{S}^{k,r}_\infty[F;Y]$ of k-contact and r-horizontal graded forms

$$\phi=\sum_{0\leq|\Lambda_i|}\phi^{\Lambda_1\ldots\Lambda_k}_{A_1\ldots A_k\mu_1\ldots\mu_r}\theta^{A_1}_{\Lambda_1}\wedge\cdots\wedge\theta^{A_k}_{\Lambda_k}\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_r}, \qquad \phi\in\mathcal{S}^{k,r}_\infty[F;Y],$$

$$h_k:\mathcal{S}^*_\infty[F;Y]\to\mathcal{S}^{k,*}_\infty[F;Y], \qquad h^r:\mathcal{S}^*_\infty[F;Y]\to\mathcal{S}^{*,r}_\infty[F;Y].$$

Accordingly, the graded exterior differential d on $\mathcal{S}^*_\infty[F;Y]$ falls into the sum $d=d_H+d_V$ of the total and vertical differentials where

$$d_H\circ h_k = h_k\circ d\circ h_k, \qquad d_H(\phi)=dx^\lambda\wedge d_\lambda(\phi), \qquad d_\lambda=\partial_\lambda+\sum_{0\leq|\Lambda|}s^A_{\lambda+\Lambda}\partial^\Lambda_A.$$

With the graded projection endomorphism

$$\varrho=\sum_{k>0}\frac{1}{k}\overline{\varrho}\circ h_k\circ h^n, \qquad \overline{\varrho}(\phi)=\sum_{0\leq|\Lambda|}(-1)^{|\Lambda|}\theta^A\wedge[d_\Lambda(\partial^\Lambda_A\rfloor\phi)], \qquad \phi\in\mathcal{S}^{>0,n}_\infty[F;Y],$$

and the graded variational operator $\delta = \varrho \circ d$, the BGDA $\mathcal{S}^*_\infty[F;Y]$ is split into the above mentioned Grassmann-graded variational bicomplex. We restrict our consideration to its short variational subcomplex and the subcomplex of one-contact graded forms

$$0 \to \mathbb{R} \longrightarrow \mathcal{S}^0_\infty[F;Y] \xrightarrow{d_H} \mathcal{S}^{0,1}_\infty[F;Y] \cdots \xrightarrow{d_H} \mathcal{S}^{0,n}_\infty[F;Y] \xrightarrow{\delta} \mathbf{E}_1 = \varrho(\mathcal{S}^{1,n}_\infty[F;Y]), \quad (2.12)$$
$$0 \to \mathcal{S}^{1,0}_\infty[F;Y] \xrightarrow{d_H} \mathcal{S}^{1,1}_\infty[F;Y] \cdots \xrightarrow{d_H} \mathcal{S}^{1,n}_\infty[F;Y] \xrightarrow{\varrho} \mathbf{E}_1 \to 0. \quad (2.13)$$

One can think of their even elements

$$L = \mathcal{L}\omega \in \mathcal{S}^{0,n}_\infty[F;Y], \qquad \omega = dx^1 \wedge \cdots \wedge dx^n, \quad (2.14)$$
$$\delta L = \theta^A \wedge \mathcal{E}_A\omega = \sum_{0\leq|\Lambda|} (-1)^{|\Lambda|}\theta^A \wedge d_\Lambda(\partial^\Lambda_A L)\omega \in \mathbf{E}_1 \quad (2.15)$$

as being a graded Lagrangian and its Euler–Lagrange operator, respectively.

Remark 2.4. Any graded density L (2.14) obeys the identity

$$0 = (\delta\circ\delta)(L) = \sum_{0\leq|\Lambda|} (-1)^{|\Lambda|}\theta^A \wedge d_\Lambda(\partial^\Lambda_A \rfloor d(\delta L)) \wedge \omega =$$
$$\sum_{0\leq|\Lambda|} [-\theta^A \wedge \theta^B_\Lambda \partial^\Lambda_B \mathcal{E}_A + (-1)^{|\Lambda|+[A][B]}\theta^B \wedge d_\Lambda(\theta^A \partial^\Lambda_B \mathcal{E}_A)] \wedge \omega = 0,$$

which leads to useful equalities

$$\eta(\partial_B \mathcal{E}_A)^\Lambda = (-1)^{[A][B]}\partial^\Lambda_A \mathcal{E}_B \quad (2.16)$$

(see the notation (2.10)). It should be noted that

$$\eta(\partial_B \mathcal{E}_A)^{\Lambda=0} = (-1)^{[A][B]}\partial_A \mathcal{E}_B + S_{AB}, \qquad S_{AB} = (-1)^{[A][B]}S_{BA},$$

but $S_{AB} = 0$ because the relation (2.11) results in

$$\eta(\eta(\partial_B \mathcal{E}_A))^{\Lambda=0} = \eta((-1)^{[A][B]}\partial_A \mathcal{E}_B + S_{AB})^{\Lambda=0} =$$
$$(-1)^{[A][B]}\eta(\partial_A \mathcal{E}_B)^{\Lambda=0} + \eta(S_{AB})^{\Lambda=0} = \partial_B \mathcal{E}_A + (-1)^{[A][B]}S_{AB} + S_{AB} = \partial_B \mathcal{E}_A.$$

If $Y \to X$ is an affine bundle, cohomology of the complex (2.12) equals the de Rham cohomology of X, while the complex (2.13) is exact [18]. Forthcoming Theorem 2.2 (see Appendix A for its proof) generalizes this result to an arbitrary fiber bundle $Y \to X$.

Theorem 2.2. *(i) Cohomology of the complex (2.12) equals the de Rham cohomology $H^*(Y)$ of Y. (ii) The complex (2.13) is exact.*

Corollary 2.3. *A δ-closed (i.e., variationally trivial) graded density $L \in \mathcal{S}^{0,n}_\infty[F;Y]$ reads*

$$L = h_0\psi + d_H\xi, \qquad \xi \in \mathcal{S}^{0,n-1}_\infty[F;Y], \quad (2.17)$$

where ψ is a closed n-form on Y. In particular, a δ-closed odd graded density is d_H-exact.

Corollary 2.4. *Any graded density L admits the decomposition*

$$dL = \delta L - d_H\Xi, \qquad \Xi \in \mathcal{S}^{1,n-1}_\infty[F;Y], \tag{2.18}$$

where $L+\Xi$ is a Lepagean equivalent of L [18].

The decomposition (2.18) leads to the first variational formula (2.20) for graded Lagrangians [5, 18]. Let $\vartheta \in \mathfrak{d}\mathcal{S}^0_\infty[F;Y]$ be a graded derivation of the $\mathbb{R}$-ring $\mathcal{S}^0_\infty[F;Y]$. The interior product $\vartheta\rfloor\phi$ and the Lie derivative $\mathbf{L}_\vartheta\phi$, $\phi \in \mathcal{S}^*_\infty[F;Y]$, are defined by the formulas

$$\begin{aligned}
&\vartheta\rfloor\phi = \vartheta^\lambda\phi_\lambda + (-1)^{[\phi_A]}\vartheta^A\phi_A, \qquad \phi \in \mathcal{S}^1_\infty[F;Y],\\
&\vartheta\rfloor(\phi\wedge\sigma) = (\vartheta\rfloor\phi)\wedge\sigma + (-1)^{|\phi|+[\phi][\vartheta]}\phi\wedge(\vartheta\rfloor\sigma), \qquad \phi,\sigma \in \mathcal{S}^*_\infty[F;Y],\\
&\mathbf{L}_\vartheta\phi = \vartheta\rfloor d\phi + d(\vartheta\rfloor\phi), \qquad \mathbf{L}_\vartheta(\phi\wedge\sigma) = \mathbf{L}_\vartheta(\phi)\wedge\sigma + (-1)^{[\vartheta][\phi]}\phi\wedge\mathbf{L}_\vartheta(\sigma).
\end{aligned}$$

A graded derivation ϑ is called contact if the Lie derivative $\mathbf{L}_\vartheta$ preserves the ideal of contact graded forms of the BGDA $\mathcal{S}^*_\infty[F;Y]$. Further, we restrict our consideration to vertical contact graded derivations, vanishing on $\mathcal{O}^*X \subset \mathcal{S}^*_\infty[F;Y]$. With respect to the local basis $\{s^A\}$ for the BGDA $\mathcal{S}^*_\infty[F;Y]$, any vertical contact graded derivation takes the form

$$\vartheta = \upsilon^A\partial_A + \sum_{0<|\Lambda|} d_\Lambda\upsilon^A\partial^\Lambda_A, \tag{2.19}$$

where the tuple of graded derivations $\{\partial^\Lambda_A\}$ is defined as the dual of the tuple of contact forms $\{\theta^A_\Lambda\}$, and υ^A [18]. Such a derivation is completely determined by its first summand $\upsilon = \upsilon^A\partial_A$, called a generalized graded vector field. It satisfies the relations

$$\vartheta\rfloor d_H\phi = -d_H(\vartheta\rfloor\phi), \qquad \mathbf{L}_\vartheta(d_H\phi) = d_H(\mathbf{L}_\vartheta\phi), \qquad \phi \in \mathcal{S}^*_\infty[F;Y].$$

Then it follows from the splitting (2.18) that the Lie derivative $\mathbf{L}_\vartheta L$ of a Lagrangian L along a vertical contact graded derivation ϑ (2.19) fulfills the first variational formula

$$\mathbf{L}_\vartheta L = \upsilon\rfloor\delta L + d_H(\vartheta\rfloor\Xi)). \tag{2.20}$$

One says that an odd vertical contact graded derivation ϑ (2.19) is a variational supersymmetry of a graded Lagrangian L if the Lie derivative $\mathbf{L}_\vartheta L$ is d_H-exact, i.e., the odd graded density $\upsilon\rfloor\delta L = \upsilon^A\mathcal{E}_A\omega$ is d_H-exact.

Remark 2.5. Given local graded functions f^Λ, $0 \le |\Lambda| \le k$, and f', there is the useful relation

$$\sum_{0\le|\Lambda|\le k} f^\Lambda d_\Lambda f'\omega = \sum_{0\le|\Lambda|\le k} (-1)^{|\Lambda|} d_\Lambda(f^\Lambda)f'\omega + d_H\sigma. \tag{2.21}$$

The first variational formula (2.20) takes the local form (2.21), but it follows from Corollary 2.3) that the second term in its right-hand side is globally d_H-exact.

A vertical contact graded derivation ϑ (2.19) is called nilpotent if $\mathbf{L}_\vartheta(\mathbf{L}_\vartheta\phi) = 0$ for any horizontal graded form $\phi \in \mathcal{S}^{0,*}_\infty[F;Y]$. One can show that ϑ (2.19) is nilpotent only if it is odd and iff all υ^A obey the equality

$$\vartheta(\upsilon) = \vartheta(\upsilon^A\partial_A) = \sum_{0\le|\Sigma|} \upsilon^B_\Sigma\partial^\Sigma_B(\upsilon^A)\partial_A = 0. \tag{2.22}$$

For the sake of simplicity, the common symbol further stands for a generalized vector field υ, the contact graded derivation ϑ (2.19) determined by υ and the Lie derivative $\mathbf{L}_\vartheta$. We agree to call all these operators the graded derivation of the BGDA $\mathcal{S}^*_\infty[F;Y]$.

Remark 2.6. We further deal with right contact graded derivations $\overleftarrow{\upsilon} = \overleftarrow{\partial}_A \upsilon^A$ of the BGDA $\mathcal{S}^*_\infty[F;Y]$ (see Remark 2.2). They act on graded forms ϕ on the right by the rule

$$\overleftarrow{\upsilon}(\phi) = \overleftarrow{\mathrm{d}}(\phi)\lfloor\overleftarrow{\upsilon} + \overleftarrow{\mathrm{d}}(\phi\lfloor\overleftarrow{\upsilon}), \qquad \overleftarrow{\upsilon}(\phi\wedge\phi') = (-1)^{[\phi'][\overleftarrow{\upsilon}]}\,\overleftarrow{\upsilon}(\phi)\wedge\phi' + \phi\wedge\overleftarrow{\upsilon}(\phi').$$

For instance, $\overleftarrow{\partial}_A(\phi) = (-1)^{([\phi]+1)[A]}\partial_A(\phi)$, $\overleftarrow{\mathrm{d}}_\Lambda = d_\Lambda$ and $\overleftarrow{\mathrm{d}}_H(\phi) = (-1)^{|\phi|}d_H(\phi)$. With right graded derivations, one obtains the right Euler–Lagrange operator

$$\overleftarrow{\delta}\, L = \overleftarrow{\mathcal{E}}_A\omega\wedge\theta^A, \qquad \overleftarrow{\mathcal{E}}_A = \sum_{0\leq|\Lambda|}(-1)^{|\Lambda|}d_\Lambda(\overleftarrow{\partial}{}^\Lambda_A(L)).$$

An odd right graded derivation $\overleftarrow{\upsilon}$ is a variational supersymmetry of a graded Lagrangian L iff the odd graded density $\overleftarrow{\mathcal{E}}_A\upsilon^A\omega$ is d_H-exact.

3 The Koszul–Tate complex

Given a graded Lagrangian L (2.14), let us associate to its Euler–Lagrange operator δL (2.15) the exact chain Koszul–Tate complex with the boundary operator whose nilpotency conditions provide the Noether and higher-stage Noether identities for δL [6]. We follow the general construction of the Koszul–Tate complex for differential operators [24].

Remark 3.1. If there is no danger of confusion, elements of homology are identified to its representatives. A chain complex is called r-exact if its homology of degree $k\leq r$ is trivial.

We start with the following notation. Given a vector bundle $E\to X$ and its pull-back E_Y onto Y, let us consider the BGDA $\mathcal{S}^*_\infty[F;E_Y]$. There are monomorphisms of $\mathcal{O}^0_\infty Y$-algebras

$$\mathcal{S}^*_\infty[F;Y]\to\mathcal{S}^*_\infty[F;E_Y], \qquad \mathcal{O}^*_\infty E\to\mathcal{S}^*_\infty[F;E_Y],$$

whose images contain the common subalgebra $\mathcal{O}^*_\infty Y$. Let us consider: (i) the subring $\mathcal{P}^0_\infty E_Y\subset\mathcal{O}^0_\infty E_Y$ of polynomial functions in fiber coordinates of the vector bundles $J^rE_Y\to J^rY$, (ii) the corresponding subring $\mathcal{P}^0_\infty[F;E_Y]\subset\mathcal{S}^0_\infty[F;E_Y]$ of graded functions with polynomial coefficients belonging to $\mathcal{P}^0_\infty E_Y$, (iii) the subalgebra $\mathcal{P}^*_\infty[F;Y;E]$ of the BGDA $\mathcal{S}^*_\infty[F;E_Y]$ over the subring $\mathcal{P}^0_\infty[F;E_Y]$. Given vector bundles V,V',E,E' over X, we further use the notation

$$\mathcal{P}^*_\infty[V'V;F;Y;EE'] = \mathcal{P}^*_\infty[V'\underset{X}{\times}V\underset{X}{\times}F;Y;E\underset{X}{\times}E']. \tag{3.1}$$

By a density-dual of a vector bundle $E\to X$ is meant $\overline{E}^* = E^*\otimes_X\overset{n}{\wedge}T^*X$.

Proposition 3.1. *The BGDA $\mathcal{P}^*_\infty[F;Y;E]$ and, similarly, the BGDA (3.1) possess the same cohomology as $\mathcal{S}^*_\infty[F;Y]$ in Theorem (2.2).*

Proof. Since $H^*(Y) = H^*(E_Y)$, this cohomology of the BGDA $\mathcal{S}^*_\infty[F;Y]$ equals that of the BGDA $\mathcal{S}^*_\infty[F;E_Y]$. Furthermore, one can replace the BGDA $\mathcal{S}^*_\infty[F;E_Y]$ with $\mathcal{P}^*_\infty[F;Y;E]$ in the condition of Theorem (2.2) due to the fact that sheaves of $\mathcal{P}^0_\infty E_Y$-modules are also sheaves of $\mathcal{O}^0_\infty Y$-modules. □

Remark 3.2. In a general setting, one must require that transition functions of fiber bundles over Y under consideration do not vanish on the shell. For the sake of simplicity, we here restrict our consideration to fiber bundles over Y which are the pull-back onto Y of fiber bundles over X. In particular, a fiber bundle $Y \to X$ of even fields is assumed to admit the vertical splitting $VY = Y \times_X W$, where W is a vector bundle over X. Let $\overline{Y}^*$ denote the density-dual of W in this splitting.

Proposition 3.2. *One can associate to a graded Lagrangian L (2.14) the chain complex (3.2) whose boundaries vanish on the shell.*

Proof. Let us extend the BGDA $\mathcal{S}^*_\infty[F;Y]$ to the BGDA $\mathcal{P}^*_\infty[\overline{Y}^*;F;Y;\overline{F}^*]$ whose local basis is $\{s^A, \overline{s}_A\}$, where $[\overline{s}_A] = ([A]+1)\mathrm{mod}\,2$. We call $\overline{s}_A$ the antifields of antifield number $\mathrm{Ant}[\overline{s}_A] = 1$. The BGDA $\mathcal{P}^*_\infty[\overline{Y}^*;F;Y;\overline{F}^*]$ is provided with the nilpotent right graded derivation $\overline{\delta} = \overleftarrow{\partial}^A \mathcal{E}_A$, where $\mathcal{E}_A$ are the graded variational derivatives (2.15). We call $\overline{\delta}$ the Koszul–Tate differential, and say that an element $\phi \in \mathcal{P}^*_\infty[\overline{Y}^*;F;Y;\overline{F}^*]$ vanishes on the shell if it is $\overline{\delta}$-exact, i.e., $\phi = \overline{\delta}\sigma$. Let us consider the module $\mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]$ of graded densities. It is split into the chain complex

$$0 \leftarrow \mathcal{S}^{0,n}_\infty[F;Y] \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_1 \cdots \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_k \cdots \tag{3.2}$$

graded by the antifield number. Its boundaries, by definition, vanish on the shell. □

Since the homology $H_{k\neq 1}(\overline{\delta})$ is not essential for our consideration, let us replace the complex (3.2) with the finite one

$$0 \leftarrow \mathrm{Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_1 \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_2. \tag{3.3}$$

It is exact at $\mathrm{Im}\,\overline{\delta}$, and its first homology coincides with that of the complex (3.2). Let us consider this homology. A generic one-chain of the complex (3.3) takes the form

$$\Phi = \sum_{0\leq|\Lambda|} \Phi^{A,\Lambda}\overline{s}_{\Lambda A}\omega, \qquad \Phi^{A,\Lambda} \in \mathcal{S}^0_\infty[F;Y], \tag{3.4}$$

and the cycle condition $\overline{\delta}\Phi = 0$ reads

$$\sum_{0\leq|\Lambda|} \Phi^{A,\Lambda} d_\Lambda \mathcal{E}_A \omega = 0. \tag{3.5}$$

One can think of this equality as being a reduction condition on the graded variational derivatives $\mathcal{E}_A$. Conversely, any reduction condition of form (3.5) comes from some cycle (3.4). The reduction condition (3.5) is trivial if a cycle is a boundary, i.e., it takes the form

$$\Phi = \sum_{0\leq|\Lambda|,|\Sigma|} T^{(A\Lambda)(B\Sigma)} d_\Sigma \mathcal{E}_B \overline{s}_{\Lambda A}\omega, \qquad T^{(A\Lambda)(B\Sigma)} = -(-1)^{[A][B]} T^{(B\Sigma)(A\Lambda)}. \tag{3.6}$$

A Lagrangian system is called degenerate if there exist non-trivial reduction conditions (3.5), called the Noether identities.

One can say something more if the $\mathcal{S}^0_\infty[F;Y]$-module $H_1(\overline{\delta})$ is finitely generated, i.e., it possesses the following particular structure. There are elements $\Delta \in H_1(\overline{\delta})$ making up a $\mathbb{Z}_2$-graded projective $C^\infty(X)$-module $\mathcal{C}_{(0)}$ of finite rank which, by virtue of Theorem 2.1, is isomorphic to the module of sections of the product $\overline{V}^* \underset{X}{\times} \overline{E}^*$ of the density-duals of some vector bundles $V \to X$ and $E \to X$. Let $\{\Delta_r\}$ be local bases for this $C^\infty(X)$-module. Every element $\Phi \in H_1(\overline{\delta})$ factorizes

$$\Phi = \sum_{0\leq|\Xi|} G^{r,\Xi} d_\Xi \Delta_r \omega, \qquad G^{r,\Xi} \in \mathcal{S}^0_\infty[F;Y], \tag{3.7}$$

$$\Delta_r = \sum_{0\leq|\Lambda|} \Delta_r^{A,\Lambda} \overline{s}_{\Lambda A}, \qquad \Delta_r^{A,\Lambda} \in \mathcal{S}^0_\infty[F;Y], \tag{3.8}$$

via elements of $\mathcal{C}_{(0)}$, i.e., any Noether identity (3.5) is a corollary of Noether identities

$$\sum_{0\leq|\Lambda|} \Delta_r^{A,\Lambda} d_\Lambda \mathcal{E}_A = 0. \tag{3.9}$$

Clearly, the factorization (3.7) is independent of specification of local bases $\{\Delta_r\}$. We say that the Noether identities (3.9) are complete, and call $\Delta_r \in \mathcal{C}_{(0)}$ the Noether operators.

Example 3.3. Let L (2.14) be a variationally trivial Lagrangian. Its Euler–Lagrange operator $\delta L = 0$ obeys the Noether identities which are finitely generated by the Noether operators $\Delta_A = \overline{s}_A$. For instance, this is the case of the topological Yang–Mills theory.

Proposition 3.3. *If the homology $H_1(\overline{\delta})$ of the complex (3.3) is finitely generated, this complex can be extended to the one-exact complex (3.11) with a boundary operator whose nilpotency conditions are equivalent to the complete Noether identities (3.9).*

Proof. Let us enlarge the BGDA $\mathcal{P}^*_\infty[\overline{Y}^*;F;Y;\overline{F}^*]$ to the BGDA

$$\mathcal{P}^*_\infty[\overline{E}^*\overline{Y}^*;F;Y;\overline{F}^*\overline{V}^*], \tag{3.10}$$

possessing the local basis $\{s^A, \overline{s}_A, \overline{c}_r\}$ where $[\overline{c}_r] = ([\Delta_r]+1)\mathrm{mod}\,2$ and $\mathrm{Ant}[\overline{c}_r] = 2$. The BGDA (3.10) is provided with the nilpotent right graded derivation $\delta_0 = \overline{\delta} + \overleftarrow{\partial}^r \Delta_r$, called the zero-stage Koszul–Tate differential. Its nilpotency conditions (2.22) are equivalent to the complete Noether identities (3.9). Then the module $\mathcal{P}^{0,n}_\infty[\overline{E}^*\overline{Y}^*;F;Y;\overline{F}^*\overline{V}^*]_{\leq 3}$ of graded densities of antifield number $\mathrm{Ant}[\phi] \leq 3$ is split into the chain complex

$$\begin{aligned} 0 \leftarrow \mathrm{Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_1 \xleftarrow{\delta_0} \mathcal{P}^{0,n}_\infty[\overline{E}^*\overline{Y}^*;F;Y;\overline{F}^*\overline{V}^*]_2 \\ \xleftarrow{\delta_0} \mathcal{P}^{0,n}_\infty[\overline{E}^*\overline{Y}^*;F;Y;\overline{F}^*\overline{V}^*]_3. \end{aligned} \tag{3.11}$$

Let $H_*(\delta_0)$ denote its homology. We have $H_0(\delta_0) = H_0(\overline{\delta}) = 0$. Furthermore, any one-cycle Φ up to a boundary takes the form (3.7) and, therefore, it is a δ_0-boundary

$$\Phi = \sum_{0\leq|\Sigma|} G^{r,\Xi} d_\Xi \Delta_r \omega = \delta_0 (\sum_{0\leq|\Sigma|} G^{r,\Xi} \overline{c}_{\Xi r} \omega).$$

Hence, $H_1(\delta_0)=0$, i.e., the complex (3.11) is one-exact. $\Box$

Turn now to the homology $H_2(\delta_0)$ of the complex (3.11). A generic two-chain reads

$$\Phi = G + H = \sum_{0\leq|\Lambda|} G^{r,\Lambda}\overline{c}_{\Lambda r}\omega + \sum_{0\leq|\Lambda|,|\Sigma|} H^{(A,\Lambda)(B,\Sigma)}\overline{s}_{\Lambda A}\overline{s}_{\Sigma B}\omega. \tag{3.12}$$

The cycle condition $\delta_0\Phi=0$ takes the form

$$\sum_{0\leq|\Lambda|} G^{r,\Lambda}d_\Lambda\Delta_r\omega + \overline{\delta}H = 0. \tag{3.13}$$

One can think of this equality as being the reduction condition on the Noether operators (3.8). Conversely, let

$$\Phi = \sum_{0\leq|\Lambda|} G^{r,\Lambda}\overline{c}_{\Lambda r}\omega \in \mathcal{P}^{0,n}_\infty[\overline{E}^*\overline{Y}^*;F;Y;\overline{F}^*\overline{V}^*]_2$$

be a graded density such that the reduction condition (3.13) holds. Obviously, this reduction condition is a cycle condition of the two-chain (3.12). The reduction condition (3.13) is trivial either if a two-cycle Φ (3.12) is a boundary or its summand G, linear in antifields, vanishes on the shell.

A degenerate Lagrangian system in Proposition 3.3 is said to be one-stage reducible if there exist non-trivial reduction conditions (3.13), called the first-stage Noether identities.

Proposition 3.4. *First-stage Noether identities can be identified to non-trivial elements of the homology $H_2(\delta_0)$ iff any $\overline{\delta}$-cycle $\phi\in\mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_2$ is a δ_0-boundary.*

Proof. It suffices to show that, if the summand G of a two-cycle Φ (3.12) is $\overline{\delta}$-exact, then Φ is a boundary. If $G=\overline{\delta}\Psi$, then

$$\Phi = \delta_0\Psi + (\overline{\delta}-\delta_0)\Psi + H. \tag{3.14}$$

The cycle condition reads

$$\delta_0\Phi = \overline{\delta}((\overline{\delta}-\delta_0)\Psi + H) = 0.$$

Then $(\overline{\delta}-\delta_0)\Psi + H$ is δ_0-exact since any $\overline{\delta}$-cycle $\phi\in\mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_2$, by assumption, is a δ_0-boundary. Consequently, Φ (3.14) is δ_0-exact. Conversely, let $\Phi\in\mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_2$ be an arbitrary $\overline{\delta}$-cycle. The cycle condition reads

$$\overline{\delta}\Phi = 2\Phi^{(A,\Lambda)(B,\Sigma)}\overline{s}_{\Lambda A}\overline{\delta}\overline{s}_{\Sigma B}\omega = 2\Phi^{(A,\Lambda)(B,\Sigma)}\overline{s}_{\Lambda A}d_\Sigma\mathcal{E}_B\omega = 0. \tag{3.15}$$

It follows that $\Phi^{(A,\Lambda)(B,\Sigma)}\overline{\delta}\overline{s}_{\Sigma B}=0$ for all indices (A,Λ). We obtain

$$\Phi^{(A,\Lambda)(B,\Sigma)}\overline{s}_{\Sigma B} = G^{(A,\Lambda)(r,\Xi)}d_\Xi\Delta_r + \overline{\delta}S^{(A,\Lambda)}.$$

Hence, Φ takes the form

$$\Phi = G'^{(A,\Lambda)(r,\Xi)}d_\Xi\Delta_r\overline{s}_{\Lambda A}\omega + \overline{\delta}S^{(A,\Lambda)}\overline{s}_{\Lambda A}\omega. \tag{3.16}$$

We can associate to Φ (3.16) the three-chain

$$\Psi = G'^{(A,\Lambda)(r,\Xi)}\overline{c}_{\Xi r}\overline{s}_{\Lambda A}\omega + S^{(A,\Lambda)}\overline{s}_{\Lambda A}\omega$$

such that

$$\delta_0\Psi = \Phi + \sigma = \Phi + G''^{(A,\Lambda)(r,\Xi)}d_\Lambda\mathcal{E}_A\overline{c}_{\Xi r}\omega + S'^{(A,\Lambda)}\overline{\delta}\overline{s}_{\Lambda A}\omega.$$

Owing to the equality $\overline{\delta}\Phi = 0$, we have $\delta_0\sigma = 0$. Since the term G'' of σ is $\overline{\delta}$-exact, then σ by assumption is δ_0-exact, i.e., $\sigma = \delta_0\psi$. It follow that $\Phi = \delta_0\Psi - \delta_0\psi$. □

If the condition of Proposition 3.4 (called the two-homology regularity condition) is satisfied, let us assume that the first-stage Noether identities are finitely generated as follows. There are elements $\Delta_{(1)} \in H_2(\delta_0)$ making up a $\mathbb{Z}_2$-graded projective $C^\infty(X)$-module $\mathcal{C}_{(1)}$ of finite rank which is isomorphic to the module of sections of the product $\overline{V}_1^* \underset{X}{\times} \overline{E}_1^*$ of the density-duals of some vector bundles $V_1 \to X$ and $E_1 \to X$. Let $\{\Delta_{r_1}\}$ be local bases for this $C^\infty(X)$-module. Every element $\Phi \in H_2(\delta_0)$ factorizes

$$\Phi = \sum_{0\leq|\Xi|} \Phi^{r_1,\Xi} d_\Xi\Delta_{r_1}\omega, \qquad \Phi^{r_1,\Xi} \in \mathcal{S}^0_\infty[F;Y], \tag{3.17}$$

$$\Delta_{r_1} = G_{r_1} + h_{r_1} = \sum_{0\leq|\Lambda|} \Delta^{r,\Lambda}_{r_1}\overline{c}_{\Lambda r} + h_{r_1}, \qquad h_{r_1}\omega \in \mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*], \tag{3.18}$$

via elements of $\mathcal{C}_{(1)}$, i.e., any first-stage Noether identity (3.13) results from the equalities

$$\sum_{0\leq|\Lambda|} \Delta^{r,\Lambda}_{r_1} d_\Lambda\Delta_r + \overline{\delta}h_{r_1} = 0, \tag{3.19}$$

called the complete first-stage Noether identities. Elements of $\mathcal{C}_{(1)}$ are called the first-stage Noether operators. Note that first summands G_{r_1} of operators Δ_{r_1} (3.18) are not $\overline{\delta}$-exact.

Proposition 3.5. *Given a reducible degenerate Lagrangian system, let the associated one-exact complex (3.11) obey the two-homology regularity condition, and let its homology $H_2(\delta_0)$ be finitely generated. Then this complex is extended to the two-exact complex (3.20) with a boundary operator whose nilpotency conditions are equivalent to complete Noether and first-stage Noether identities.*

Proof. Let us consider the BGDA $\mathcal{P}^*_\infty[\overline{E}_1^*\overline{E}^*\overline{Y}^*;F;Y;\overline{F}^*\overline{V}^*\overline{V}_1^*]$ possessing the local basis $\{s^A, \overline{s}_A, \overline{c}_r, \overline{c}_{r_1}\}$, where $[\overline{c}_{r_1}] = ([\Delta_{r_1}]+1)\mathrm{mod}\,2$ and $\mathrm{Ant}[\overline{c}_{r_1}] = 3$. It can be provided with the nilpotent graded derivation $\delta_1 = \delta_0 + \overleftarrow{\partial}^{r_1}\Delta_{r_1}$, called the first-stage Koszul–Tate differential. Its nilpotency conditions (2.22) are equivalent to the complete Noether identities (3.9) and complete first-stage Noether identities (3.19). Then the module $\mathcal{P}^{0,n}_\infty[\overline{E}_1^*\overline{E}^*\overline{Y}^*;F;Y;\overline{F}^*\overline{V}^*\overline{V}_1^*]_{\leq 4}$ of graded densities of antifield number $\mathrm{Ant}[\phi] \leq 4$ is split into the chain complex

$$\begin{aligned} 0 \leftarrow \mathrm{Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_1 \xleftarrow{\delta_0} \mathcal{P}^{0,n}_\infty[\overline{E}^*\overline{Y}^*;F;Y;\overline{F}^*\overline{V}^*]_2 \xleftarrow{\delta_1} \\ \mathcal{P}^{0,n}_\infty[\overline{E}_1^*\overline{E}^*\overline{Y}^*;F;Y;\overline{F}^*\overline{V}^*\overline{V}_1^*]_3 \xleftarrow{\delta_1} \mathcal{P}^{0,n}_\infty[\overline{E}_1^*\overline{E}^*\overline{Y}^*;F;Y;\overline{F}^*\overline{V}^*\overline{V}_1^*]_4. \end{aligned} \tag{3.20}$$

Let $H_*(\delta_1)$ denote its homology. It is readily observed that

$$H_0(\delta_1) = H_0(\overline{\delta}), \qquad H_1(\delta_1) = H_1(\delta_0) = 0.$$

By virtue of the expression (3.17), any two-cycle of the complex (3.20) is a boundary

$$\Phi = \sum_{0\leq|\Xi|} \Phi^{r_1,\Xi} d_\Xi \Delta_{r_1}\omega = \delta_1(\sum_{0\leq|\Xi|} \Phi^{r_1,\Xi}\overline{c}_{\Xi r_1})\omega.$$

It follows that $H_2(\delta_1) = 0$, i.e., the complex (3.20) is two-exact. $\Box$

If the third homology $H_3(\delta_1)$ of the complex (3.20) is not trivial, there are reduction conditions on the first-stage Noether operators, and so on.

Iterating the arguments, we come to the following. Let $(\mathcal{S}^*_\infty[F;Y], L)$ be a degenerate Lagrangian system whose Noether identities are finitely generated. In accordance with Proposition 3.3, we associates to it the one-exact chain complex (3.11). Given an integer $N \geq 1$, let $V_1, \ldots, V_N, E_1, \ldots, E_N$ be some vector bundles over X and

$$\overline{\mathcal{P}}^*_\infty\{N\} = \mathcal{P}^*_\infty[\overline{E}^*_N \cdots \overline{E}^*_1 \overline{E}^* \overline{Y}^*; F; Y; \overline{F}^* \overline{V}^* \overline{V}^*_1 \cdots \overline{V}^*_N] \tag{3.21}$$

the BGDA with the local basis $\{s^A, \overline{s}_A, \overline{c}_r, \overline{c}_{r_1}, \ldots, \overline{c}_{r_N}\}$ graded by antifield numbers $\mathrm{Ant}[\overline{c}_{r_k}] = k+2$. Let $k = -1, 0$ further stand for $\overline{s}_A$ and $\overline{c}_r$, respectively. We assume that the BGDA $\overline{\mathcal{P}}^*_\infty\{N\}$ (3.21) is provided with a nilpotent graded derivation

$$\delta_N = \delta_0 + \sum_{1\leq k\leq N} \overleftarrow{\partial}^{r_k} \Delta_{r_k}, \tag{3.22}$$

$$\Delta_{r_k} = G_{r_k} + h_{r_k} = \sum_{0\leq|\Lambda|} \Delta^{r_{k-1},\Lambda}_{r_k} \overline{c}_{\Lambda r_{k-1}} + \sum_{0\leq|\Sigma|,|\Xi|} (h^{(r_{k-2},\Sigma)(A,\Xi)}_{r_k} \overline{c}_{\Sigma r_{k-2}} \overline{s}_{\Xi A} + ...), \tag{3.23}$$

of antifield number -1, and that the module $\overline{\mathcal{P}}^{0,n}_\infty\{N\}_{\leq N+3}$ of graded densities of antifield number $\mathrm{Ant}[\phi] \leq N+3$ is split into the $(N+1)$-exact chain complex

$$\begin{aligned} 0 \leftarrow \mathrm{Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{Y}^*; F; Y; \overline{F}^*]_1 \xleftarrow{\delta_0} \overline{\mathcal{P}}^{0,n}_\infty\{0\}_2 \xleftarrow{\delta_1} \overline{\mathcal{P}}^{0,n}_\infty\{1\}_3 \cdots \\ \xleftarrow{\delta_{N-1}} \overline{\mathcal{P}}^{0,n}_\infty\{N-1\}_{N+1} \xleftarrow{\delta_N} \overline{\mathcal{P}}^{0,n}_\infty\{N\}_{N+2} \xleftarrow{\delta_N} \overline{\mathcal{P}}^{0,n}_\infty\{N\}_{N+3}, \end{aligned} \tag{3.24}$$

which satisfies the following $(N+1)$-homology regularity condition.

Definition 3.6. *One says that the complex (3.24) obeys the $(N+1)$-homology regularity condition if any $\delta_{k<N-1}$-cycle $\phi \in \overline{\mathcal{P}}^{0,n}_\infty\{k\}_{k+3} \subset \overline{\mathcal{P}}^{0,n}_\infty\{k+1\}_{k+3}$ is a δ_{k+1}-boundary.*

Note that the $(N+1)$-exactness of the complex (3.24) implies that any $\delta_{k<N-1}$-cycle $\phi \in \overline{\mathcal{P}}^{0,n}_\infty\{k\}_{k+3}$, $k < N$, is a δ_{k+2}-boundary, but not necessary a δ_{k+1}-boundary.

If $N = 1$, the complex $\overline{\mathcal{P}}^{0,n}_\infty\{1\}_{\leq 4}$ (3.24) restarts the complex (3.20). Therefore, we agree to call δ_N (3.22) the N-stage Koszul–Tate differential. Its nilpotency implies the complete Noether identities (3.9), the first-stage Noether identities (3.19), and the equalities

$$\sum_{0\leq|\Lambda|} \Delta^{r_{k-1},\Lambda}_{r_k} d_\Lambda(\sum_{0\leq|\Sigma|} \Delta^{r_{k-2},\Sigma}_{r_{k-1}} \overline{c}_{\Sigma r_{k-2}}) + \overline{\delta}(\sum_{0\leq|\Sigma|,|\Xi|} h^{(r_{k-2},\Sigma)(A,\Xi)}_{r_k} \overline{c}_{\Sigma r_{k-2}} \overline{s}_{\Xi A}) = 0 \tag{3.25}$$

for $k=2,\ldots,N$. One can think of the equalities (3.25) as being complete k-stage Noether identities because of their properties which we will justify in the case of $k=N+1$. Accordingly, Δ_{r_k} (3.23) are said to be the k-stage Noether operators.

Let us consider the $(N+2)$-homology of the complex (3.24). A generic $(N+2)$-chain $\Phi\in\overline{\mathcal{P}}^{0,n}_\infty\{N\}_{N+2}$ takes the form

$$\Phi=G+H=\sum_{0\leq|\Lambda|}G^{r_N,\Lambda}\overline{c}_{\Lambda r_N}\omega+\sum_{0\leq|\Sigma|,|\Xi|}(H^{(A,\Xi)(r_{N-1},\Sigma)}\overline{s}_{\Xi A}\overline{c}_{\Sigma r_{N-1}}+...)\omega. \tag{3.26}$$

Let it be a cycle. The cycle condition $\delta_N\Phi=0$ implies the equality

$$\sum_{0\leq|\Lambda|}G^{r_N,\Lambda}d_\Lambda(\sum_{0\leq|\Sigma|}\Delta^{r_{N-1},\Sigma}_{r_N}\overline{c}_{\Sigma r_{N-1}})+\overline{\delta}(\sum_{0\leq|\Sigma|,|\Xi|}H^{(A,\Xi)(r_{N-1},\Sigma)}\overline{s}_{\Xi A}\overline{c}_{\Sigma r_{N-1}})=0. \tag{3.27}$$

One can think of this equality as being the reduction condition on the N-stage Noether operators (3.23). Conversely, let

$$\Phi=\sum_{0\leq|\Lambda|}G^{r_N,\Lambda}\overline{c}_{\Lambda r_N}\omega\in\overline{\mathcal{P}}^{0,n}_\infty\{N\}_{N+2}$$

be a graded density such that the reduction condition (3.27) holds. Then this reduction condition can be extended to a cycle one as follows. It is brought into the form

$$\begin{aligned}&\delta_N(\sum_{0\leq|\Lambda|}G^{r_N,\Lambda}\overline{c}_{\Lambda r_N}+\sum_{0\leq|\Sigma|,|\Xi|}H^{(A,\Xi)(r_{N-1},\Sigma)}\overline{s}_{\Xi A}\overline{c}_{\Sigma r_{N-1}})=\\&\quad-\sum_{0\leq|\Lambda|}G^{r_N,\Lambda}d_\Lambda h_{r_N}+\sum_{0\leq|\Sigma|,|\Xi|}H^{(A,\Xi)(r_{N-1},\Sigma)}\overline{s}_{\Xi A}d_\Sigma\Delta_{r_{N-1}}.\end{aligned}$$

A glance at the expression (3.23) shows that the term in the right-hand side of this equality belongs to $\overline{\mathcal{P}}^{0,n}_\infty\{N-2\}_{N+1}$. It is a δ_{N-2}-cycle and, consequently, a δ_{N-1}-boundary $\delta_{N-1}\Psi$ in accordance with the $(N+1)$-homology regularity condition. Then the reduction condition (3.27) is a $\overline{c}_{\Sigma r_{N-1}}$-dependent part of the cycle condition

$$\delta_N(\sum_{0\leq|\Lambda|}G^{r_N,\Lambda}\overline{c}_{\Lambda r_N}+\sum_{0\leq|\Sigma|,|\Xi|}H^{(A,\Xi)(r_{N-1},\Sigma)}\overline{s}_{\Xi A}\overline{c}_{\Sigma r_{N-1}}-\Psi)=0,$$

but $\delta_N\Psi$ does not make a contribution to this reduction condition.

Being a cycle condition, the reduction condition (3.27) is trivial either if a cycle Φ (3.26) is a δ_N-boundary or its summand G is $\overline{\delta}$-exact. Then a degenerate Lagrangian system is said to be $(N+1)$-stage reducible if there exist non-trivial reduction conditions (3.27), called the $(N+1)$-stage Noether identities.

Theorem 3.7. *(i) The $(N+1)$-stage Noether identities can be identified to non-trivial elements of the homology $H_{N+2}(\delta_N)$ of the complex (3.24) iff this homology obeys the $(N+2)$-homology regularity condition. (ii) If the homology $H_{N+2}(\delta_N)$ is finitely generated, the complex (3.24) admits an $(N+2)$-exact extension.*

Proof. (i) The $(N+2)$-homology regularity condition implies that any δ_{N-1}-cycle $\Phi\in\overline{\mathcal{P}}^{0,n}_\infty\{N-1\}_{N+2}\subset\overline{\mathcal{P}}^{0,n}_\infty\{N\}_{N+2}$ is a δ_N-boundary. Therefore, if Φ (3.26) is a representative of

a non-trivial element of $H_{N+2}(\delta_N)$, its summand G linear in $\overline{c}_{\Lambda r_N}$ does not vanish. Moreover, it is not a $\overline{\delta}$-boundary. Indeed, if $G = \overline{\delta}\Psi$, then

$$\Phi = \delta_N \Psi + (\overline{\delta} - \delta_N)\Psi + H. \tag{3.28}$$

The cycle condition takes the form

$$\delta_N \Phi = \delta_{N-1}((\overline{\delta} - \delta_N)\Psi + H) = 0.$$

Hence, $(\overline{\delta}-\delta_N)\Psi+H$ is δ_N-exact since any δ_{N-1}-cycle $\phi \in \overline{\mathcal{P}}^{0,n}_\infty\{N-1\}_{N+2}$ is a δ_N-boundary. Consequently, Φ (3.28) is a boundary. If the $(N+2)$-homology regularity condition does not hold, trivial reduction conditions (3.27) also come from non-trivial elements of the homology $H_{N+2}(\delta_N)$. (ii) Let the $(N+1)$-stage Noether identities be finitely generated. Namely, there exist elements $\Delta_{(N+1)} \in H_{N+2}(\delta_N)$ making up a $\mathbb{Z}_2$-graded projective $C^\infty(X)$-module $\mathcal{C}_{(N+1)}$ of finite rank which is isomorphic to the module of sections of the product $\overline{V}^*_{N+1} \underset{X}{\times} \overline{E}^*_{N+1}$ of the density-duals of some vector bundles $V_{N+1} \to X$ and $E_{N+1} \to X$. Let $\{\Delta_{r_{N+1}}\}$ be local bases for this $C^\infty(X)$-module. Then any element $\Phi \in H_{N+2}(\delta_N)$ factorizes

$$\Phi = \sum_{0\leq|\Xi|} \Phi^{r_{N+1},\Xi} d_\Xi \Delta_{r_{N+1}}\omega, \qquad \Phi^{r_{N+1},\Xi} \in \mathcal{S}^0_\infty[F;Y], \tag{3.29}$$

$$\Delta_{r_{N+1}} = G_{r_{N+1}} + h_{r_{N+1}} = \sum_{0\leq|\Lambda|} \Delta^{r_N,\Lambda}_{r_{N+1}} \overline{c}_{\Lambda r_N} + h_{r_{N+1}}, \tag{3.30}$$

via elements of $\mathcal{C}_{(N+1)}$. Clearly, this factorization is independent of specification of local bases $\{\Delta_{r_{N+1}}\}$. Let us extend the BGDA $\overline{\mathcal{P}}^*_\infty\{N\}$ (3.21) to the BGDA $\overline{\mathcal{P}}^*_\infty\{N+1\}$ possessing the local basis $\{s^A, \overline{s}_A, \overline{c}_r, \overline{c}_{r_1}, \ldots, \overline{c}_{r_N}, \overline{c}_{r_{N+1}}\}$ where $\mathrm{Ant}[\overline{c}_{r_{N+1}}] = N+3$ and $[\overline{c}_{r_{N+1}}] = ([\Delta_{r_{N+1}}] + 1) \bmod 2$. It is provided with the nilpotent graded derivation $\delta_{N+1} = \delta_N + \overleftarrow{\partial}^{r_{N+1}} \Delta_{r_{N+1}}$ of antifield number -1. With this graded derivation, the module $\overline{\mathcal{P}}^{0,n}_\infty\{N+1\}_{\leq N+4}$ of graded densities of antifield number $\mathrm{Ant}[\phi] \leq N+4$ is split into the chain complex

$$\begin{aligned} 0 \leftarrow \mathrm{Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_1 \xleftarrow{\delta_0} \overline{\mathcal{P}}^{0,n}_\infty\{0\}_2 \xleftarrow{\delta_1} \overline{\mathcal{P}}^{0,n}_\infty\{1\}_3 \cdots \\ \xleftarrow{\delta_{N-1}} \overline{\mathcal{P}}^{0,n}_\infty\{N-1\}_{N+1} \xleftarrow{\delta_N} \overline{\mathcal{P}}^{0,n}_\infty\{N\}_{N+2} \xleftarrow{\delta_{N+1}} \overline{\mathcal{P}}^{0,n}_\infty\{N+1\}_{N+3} \xleftarrow{\delta_{N+1}} \overline{\mathcal{P}}^{0,n}_\infty\{N+1\}_{N+4}. \end{aligned} \tag{3.31}$$

It is readily observed that this complex is $(N+2)$-exact. In this case, the $(N+1)$-stage Noether identities (3.27) come from the complete $(N+1)$-stage Noether identities

$$\sum_{0\leq|\Lambda|} \Delta^{r_N,\Lambda}_{r_{N+1}} d_\Lambda \Delta_{r_N}\omega + \overline{\delta} h_{r_{N+1}}\omega = 0, \tag{3.32}$$

which are reproduced as the nilpotency conditions of the graded derivation δ_{N+1}. □

The iteration procedure based on Theorem 3.7 may be infinite. We restrict our consideration to the case of a finitely (N-stage) reducible Lagrangian system possessing the finite $(N+2)$-exact Koszul–Tate complex

$$\begin{aligned} 0 \leftarrow \mathrm{Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_1 \xleftarrow{\delta_0} \overline{\mathcal{P}}^{0,n}_\infty\{0\}_2 \xleftarrow{\delta_1} \overline{\mathcal{P}}^{0,n}_\infty\{1\}_3 \cdots \\ \xleftarrow{\delta_{N-1}} \overline{\mathcal{P}}^{0,n}_\infty\{N-1\}_{N+1} \xleftarrow{\delta_N} \overline{\mathcal{P}}^{0,n}_\infty\{N\}_{N+2} \xleftarrow{\delta_N} \overline{\mathcal{P}}^{0,n}_\infty\{N\}_{N+3}, \end{aligned} \tag{3.33}$$

$$\delta_N = \overleftarrow{\partial}^A \mathcal{E}_A + \sum_{0\leq|\Lambda|} \overleftarrow{\partial}^r \Delta^{A,\Lambda}_r \overline{s}_{\Lambda A} + \sum_{1\leq k\leq N} \overleftarrow{\partial}^{r_k} \Delta_{r_k}, \tag{3.34}$$

where Δ_{r_k} are the k-stage Noether operators (3.23) which obey the Noether identities (3.9), first-stage Noether identities (3.19) and k-stage Noether identities (3.25), $k = 2, \ldots, N$. Forthcoming Theorem 4.1 associates to the Koszul–Tate complex (3.33) the sequence (4.4), graded in ghosts, whose ascent operator υ_e (4.5) provides the gauge and higher-stage gauge supersymmetries of an original graded Lagrangian.

4 The Noether second theorem

Given the BGDA $\overline{\mathcal{P}}^*_\infty\{N\}$ (3.21), let us consider the BGDA

$$\mathcal{P}^*_\infty\{N\} = \mathcal{P}^*_\infty[V_N \cdots V_1 V; F; Y; E E_1 \cdots E_N] \tag{4.1}$$

with the local basis $\{s^A, c^r, c^{r_1}, \ldots, c^{r_N}\}$ and the BGDA

$$P^*_\infty\{N\} = \mathcal{P}^*_\infty[\overline{E}^*_N \cdots \overline{E}^*_1 \overline{E}^* \overline{Y}^* V_N \cdots V_1 V; F; Y; E E_1 \cdots E_N \overline{F}^* \overline{V}^* \overline{V}^*_1 \cdots \overline{V}^*_N] \tag{4.2}$$

with the local basis

$$\{s^A, c^r, c^{r_1}, \ldots, c^{r_N}, \overline{s}_A, \overline{c}_r, \overline{c}_{r_1}, \ldots, \overline{c}_{r_N}\}, \tag{4.3}$$

where $[c^{r_k}] = ([\overline{c}_{r_k}] + 1) \mathrm{mod}\, 2$ and $\mathrm{Ant}[c^{r_k}] = -(k+1)$. We call c^{r_k}, $k = 0, \ldots, N$, the ghosts of ghost number $\mathrm{gh}[c^{r_k}] = k + 1$. Clearly, the BGDAs $\overline{\mathcal{P}}^*_\infty\{N\}$ (3.21) and $\mathcal{P}^*_\infty\{N\}$ (4.1) are subalgebras of the BGDA $P^*_\infty\{N\}$ (4.2). The Koszul–Tate differential δ_N (3.34) is naturally extended to a graded derivation of the BGDA $P^*_\infty\{N\}$ (4.2).

Theorem 4.1. *With the Koszul–Tate complex (3.33) of antifields, the graded commutative ring $\mathcal{P}^0_\infty\{N\} \subset \mathcal{P}^*_\infty\{N\}$ (4.1) of ghosts is split into the sequence*

$$0 \to \mathcal{S}^0_\infty[F;Y] \xrightarrow{u_e} \mathcal{P}^0_\infty\{N\}_1 \xrightarrow{u_e} \mathcal{P}^0_\infty\{N\}_2 \xrightarrow{u_e} \cdots, \tag{4.4}$$

$$u_e = u + \sum_{1 \leq k \leq N} u_{(k)}, \tag{4.5}$$

where u (4.11), $u_{(1)}$ (4.13) and $u_{(k)}$ (4.15), $k = 2, \ldots, N$, are the operators of gauge and higher-stage gauge supersymmetries of an original graded Lagrangian.

Proof. Let us extend an original graded Lagrangian L to the even graded density

$$L_e = \mathcal{L}_e \omega = L + L_1 = L + \sum_{0 \leq k \leq N} c^{r_k} \Delta_{r_k} \omega = L + \delta_N\Big(\sum_{0 \leq k \leq N} c^{r_k} \overline{c}_{r_k} \omega\Big), \tag{4.6}$$

whose summand L_1 is linear in ghosts and possesses the zero antifield number. It is readily observed that $\delta_N(L_e) = 0$, i.e., δ_N is a variational supersymmetry of the graded Lagrangian L_e (4.6). It follows that

$$\begin{aligned}
&\Big[\frac{\overleftarrow{\delta} \mathcal{L}_e}{\delta \overline{s}_A} \mathcal{E}_A + \sum_{0 \leq k \leq N} \frac{\overleftarrow{\delta} \mathcal{L}_e}{\delta \overline{c}_{r_k}} \Delta_{r_k}\Big]\omega = \Big[\frac{\overleftarrow{\delta} \mathcal{L}_e}{\delta \overline{s}_A} \mathcal{E}_A + \sum_{0 \leq k \leq N} \frac{\overleftarrow{\delta} \mathcal{L}_e}{\delta \overline{c}_{r_k}} \frac{\delta \mathcal{L}_e}{\delta c^{r_k}}\Big]\omega = \\
&\qquad \Big[\upsilon^A \mathcal{E}_A + \sum_{0 \leq k \leq N} \upsilon^{r_k} \frac{\delta \mathcal{L}_e}{\delta c^{r_k}}\Big]\omega = d_H \sigma,
\end{aligned} \tag{4.7}$$

$$\upsilon^A = \frac{\overleftarrow{\delta}\mathcal{L}_e}{\delta\overline{s}_A} = u^A + w^A = \sum_{0\leq|\Lambda|} c^r_\Lambda \eta(\Delta^A_r)^\Lambda + \sum_{i>0}\sum_{0\leq|\Lambda|} c^{r_i}_\Lambda \eta(\overleftarrow{\partial}^A(h_{r_i}))^\Lambda,$$

$$\upsilon^{r_k} = \frac{\overleftarrow{\delta}\mathcal{L}_e}{\delta\overline{c}_{r_k}} = u^{r_k} + w^{r_k} = \sum_{0\leq|\Lambda|} c^{r_{k+1}}_\Lambda \eta(\Delta^{r_k}_{r_{k+1}})^\Lambda + \sum_{i>k+1}\sum_{0\leq|\Lambda|} c^{r_i}_\Lambda \eta(\overleftarrow{\partial}^{r_k}(h_{r_i}))^\Lambda,$$

(see the formulas (2.9) – (2.10)). The equality (4.7) falls into the set of equalities

$$\frac{\overleftarrow{\delta}(c^r\Delta_r)}{\delta\overline{s}_A}\mathcal{E}_A\omega = u^A\mathcal{E}_A\omega = d_H\sigma_0, \tag{4.8}$$

$$[\frac{\overleftarrow{\delta}(c^{r_1}\Delta_{r_1})}{\delta\overline{s}_A}\mathcal{E}_A + \frac{\overleftarrow{\delta}(c^{r_1}\Delta_{r_1})}{\delta\overline{c}_r}\Delta_r]\omega = d_H\sigma_1, \tag{4.9}$$

$$[\frac{\overleftarrow{\delta}(c^{r_i}\Delta_{r_i})}{\delta\overline{s}_A}\mathcal{E}_A + \sum_{k<i}\frac{\overleftarrow{\delta}(c^{r_i}\Delta_{r_i})}{\delta\overline{c}_{r_k}}\Delta_{r_k}]\omega = d_H\sigma_i, \qquad i=2,\ldots,N, \tag{4.10}$$

with respect to the polynomial degree in ghosts. A glance at the equality (4.8) shows that, by virtue of the first variational formula (2.20), the graded derivation

$$u = u^A\frac{\partial}{\partial s^A}, \qquad u^A = \sum_{0\leq|\Lambda|} c^r_\Lambda \eta(\Delta^A_r)^\Lambda, \tag{4.11}$$

is a variational supersymmetry of an original graded Lagrangian L. This variational supersymmetry is parameterized by ghosts c^r. Therefore, one can think of it as being a gauge supersymmetry of L [5, 18]. The equality (4.9) takes the form

$$[\frac{\overleftarrow{\delta}}{\delta\overline{s}_A}(c^{r_1}h^{(B,\Sigma)(A,\Xi)}_{r_1}\overline{s}_{\Sigma B}\overline{s}_{\Xi A})\mathcal{E}_A + \frac{\overleftarrow{\delta}}{\delta\overline{c}_r}(c^{r_1}\sum_{0\leq|\Sigma|}\Delta^{r,\Sigma}_{r_1}\overline{c}_{\Sigma r})\sum_{0\leq|\Xi|}\Delta^{B,\Xi}_r\overline{s}_{\Xi B}]\omega =$$
$$[\sum_{0\leq|\Xi|}(-1)^{|\Xi|}d_\Xi(c^{r_1}\sum_{0\leq|\Sigma|}2h^{(B,\Sigma)(A,\Xi)}_{r_1}\overline{s}_{\Sigma B})\mathcal{E}_A + u^r\sum_{0\leq|\Xi|}\Delta^{B,\Xi}_r\overline{s}_{\Xi B}]\omega = d_H\sigma'_1.$$

Using the relation (2.21), we obtain

$$[\sum_{0\leq|\Xi|}c^{r_1}\sum_{0\leq|\Sigma|}2h^{(B,\Sigma)(A,\Xi)}_{r_1}\overline{s}_{\Sigma B}d_\Xi\mathcal{E}_A + u^r\sum_{0\leq|\Xi|}\Delta^{B,\Xi}_r\overline{s}_{\Xi B}]\omega = d_H\sigma_1.$$

The variational derivative of the both sides of this equality with respect to the antifield $\overline{s}_B$ leads to the relation

$$\sum_{0\leq|\Sigma|}\eta(h^{(B)(A,\Xi)}_{r_1})^\Sigma d_\Sigma(2c^{r_1}d_\Xi\mathcal{E}_A) + \sum_{0\leq|\Sigma|}u^r_\Sigma\eta(\Delta^B_r)^\Sigma = 0,$$

which is brought into the form

$$\sum_{0\leq|\Sigma|}d_\Sigma u^r\frac{\partial}{\partial c^r_\Sigma}u^B = \overline{\delta}(\alpha^B), \qquad \alpha^B = -\sum_{0\leq|\Sigma|}\eta(2h^{(B)(A,\Xi)}_{r_1})^\Sigma d_\Sigma(c^{r_1}\overline{s}_{\Xi A}). \tag{4.12}$$

Therefore, the odd graded derivation

$$u_{(1)} = u^r \frac{\partial}{\partial c^r}, \qquad u^r = \sum_{0\leq|\Lambda|} c_\Lambda^{r_1} \eta(\Delta_{r_1}^r)^\Lambda, \tag{4.13}$$

is the first-stage gauge supersymmetry of a reducible Lagrangian system [5]. Every equality (4.10) is split into a set of equalities with respect to the polynomial degree in antifields. Let us consider the one, linear in antifields $\overline{c}_{r_{i-2}}$ and their jets. We have

$$\begin{aligned}&[\frac{\overleftarrow{\delta}}{\delta \overline{s}_A}(c^{r_i} \sum_{0\leq|\Sigma|,|\Xi|} h_{r_i}^{(r_{i-2},\Sigma)(A,\Xi)} \overline{c}_{\Sigma r_{i-2}} \overline{s}_{\Xi A})\mathcal{E}_A + \\ &\qquad \frac{\overleftarrow{\delta}}{\delta \overline{c}_{r_{i-1}}}(c^{r_i} \sum_{0\leq|\Sigma|} \Delta_{r_i}^{r'_{i-1},\Sigma} \overline{c}_{\Sigma r'_{i-1}}) \sum_{0\leq|\Xi|} \Delta_{r_{i-1}}^{r_{i-2},\Xi} \overline{c}_{\Xi r_{i-2}}]\omega = d_H\sigma_i.\end{aligned}$$

It is brought into the form

$$[\sum_{0\leq|\Xi|}(-1)^{|\Xi|} d_\Xi(c^{r_i} \sum_{0\leq|\Sigma|} h_{r_i}^{(r_{i-2},\Sigma)(A,\Xi)} \overline{c}_{\Sigma r_{i-2}})\mathcal{E}_A + u^{r_{i-1}} \sum_{0\leq|\Xi|} \Delta_{r_{i-1}}^{r_{i-2},\Xi} \overline{c}_{\Xi r_{i-2}}]\omega = d_H\sigma_i.$$

Using the relation (2.21), we obtain

$$[\sum_{0\leq|\Xi|} c^{r_i} \sum_{0\leq|\Sigma|} h_{r_i}^{(r_{i-2},\Sigma)(A,\Xi)} \overline{c}_{\Sigma r_{i-2}} d_\Xi \mathcal{E}_A + u^{r_{i-1}} \sum_{0\leq|\Xi|} \Delta_{r_{i-1}}^{r_{i-2},\Xi} \overline{c}_{\Xi r_{i-2}}]\omega = d_H\sigma'_i.$$

The variational derivative of the both sides of this equality with respect to the antifield $\overline{c}_{r_{i-2}}$ leads to the relation

$$\sum_{0\leq|\Sigma|} \eta(h_{r_i}^{(r_{i-2})(A,\Xi)})^\Sigma d_\Sigma(c^{r_i} d_\Xi \mathcal{E}_A) + \sum_{0\leq|\Sigma|} u_\Sigma^{r_{i-1}} \eta(\Delta_{r_{i-1}}^{r_{i-2}})^\Sigma = 0,$$

which takes the form

$$\sum_{0\leq|\Sigma|} d_\Sigma u^{r_{i-1}} \frac{\partial}{\partial c_\Sigma^{r_{i-1}}} u^{r_{i-2}} = \overline{\delta}(\alpha^{r_{i-2}}), \qquad \alpha^{r_{i-2}} = -\sum_{0\leq|\Sigma|} \eta(h_{r_i}^{(r_{i-2})(A,\Xi)})^\Sigma d_\Sigma(c^{r_i} \overline{s}_{\Xi A}). \tag{4.14}$$

Therefore, the odd graded derivations

$$u_{(k)} = u^{r_{k-1}} \frac{\partial}{\partial c^{r_{k-1}}}, \qquad u^{r_{k-1}} = \sum_{0\leq|\Lambda|} c_\Lambda^{r_k} \eta(\Delta_{r_k}^{r_{k-1}})^\Lambda, \qquad k = 2, \ldots, N, \tag{4.15}$$

are the k-stage gauge supersymmetries [5]. The graded derivations u (4.11), $u_{(1)}$ (4.13), $u_{(k)}$ (4.15) are assembled into the ascent operator (4.5) of ghost number 1, that we agree to call the total gauge operator. It provides the sequence (4.4). □

The total gauge operator (4.5) need not be nilpotent even on the shell. We say that gauge and higher-stage gauge supersymmetries of a Lagrangian system form an algebra on the shell if the graded derivation (4.5) can be extended to a graded derivation υ^0 of ghost

number 1 by means of terms of higher polynomial degree in ghosts such that υ^0 is nilpotent on the shell. Namely, we have

$$\upsilon^0 = u_e + \xi = u^A\partial_A + \sum_{1\leq k\leq N}(u^{r_{k-1}} + \xi^{r_{k-1}})\partial_{r_{k-1}}, \tag{4.16}$$

where all the coefficients $\xi^{r_{k-1}}$ are at least quadratic in ghosts and $(\upsilon^0\circ\upsilon^0)(f)$ is $\overline{\delta}$-exact for any graded function $f\in\mathcal{P}^0_\infty\{N\}\subset P^0_\infty\{N\}$. This nilpotency condition falls into a set of equalities with respect to the polynomial degree in ghosts. Let us write the first and second of them involving the coefficients $\xi^{r_{k-1}}_2$ quadratic in ghosts. We have

$$\sum_{0\leq|\Sigma|} d_\Sigma u^r\partial^\Sigma_r u^B = \overline{\delta}(\alpha^B_1), \qquad \sum_{0\leq|\Sigma|} d_\Sigma u^{r_{k-1}}\partial^\Sigma_{r_{k-1}} u^{r_{k-2}} = \overline{\delta}(\alpha^{r_{k-2}}_1), \quad 2\leq k\leq N, \tag{4.17}$$

$$\sum_{0\leq|\Sigma|}[d_\Sigma u^A\partial^\Sigma_A u^B + d_\Sigma\xi^r_2\partial^\Sigma_r u^B] = \overline{\delta}(\alpha^B_2), \tag{4.18}$$

$$\sum_{0\leq|\Sigma|}[d_\Sigma u^A\partial^\Sigma_A u^{r_{k-1}} + d_\Sigma\xi^{r_k}_2\partial^\Sigma_{r_k}u^{r_{k-1}} + d_\Sigma u^{r'_{k-1}}\partial^\Sigma_{r'_{k-1}}\xi^{r_{k-1}}_2] = \overline{\delta}(\alpha^{r_{k-1}}_2), \tag{4.19}$$

$$\xi^r_2 = \xi^{r,\Lambda,\Sigma}_{r',r''}c^{r'}_\Lambda c^{r''}_\Sigma, \qquad \xi^{r_k}_2 = \xi^{r_k,\Lambda,\Sigma}_{r,r'_k}c^r_\Lambda c^{r'_k}_\Sigma, \qquad 2\leq k\leq N. \tag{4.20}$$

The equalities (4.17) reproduce the relations (4.12) and (4.14) in Theorem 4.1. The equalities (4.18) – (4.19) provide the generalized commutation relations on the shell between gauge and higher-stage gauge supersymmetries, and one can think of the coefficients ξ_2 (4.20) as being *sui generis* generalized structure functions [5, 14].

Note that the total gauge operator in an irreducible gauge theory is the operator of infinitesimal gauge transformations whose parameter functions are replaced with the ghosts. Its nilpotent extension is the familiar BRST operator [7, 18]. For instance, let $P\to X$ be a principal bundle with a structure Lie group G, whose Lie algebra possesses the basis $\{e_r\}$ and the structure constants c^r_{pq}. Let us consider a gauge theory of principal connections on P. It is an irreducible degenerate Lagrangian system. Principal connections on P are represented by sections of the quotient $C = J^1P/G$, called the bundle of principal connections. It is an affine bundle coordinated by (x^λ, a^r_λ) such that, given a section A of $C\to X$, its components $A^r_\lambda = a^r_\lambda\circ A$ are coefficients of the local connection form (i.e., gauge potentials). Infinitesimal generators of one-parameter groups of automorphisms of a principal bundle P are G-invariant projectable vector fields on P. They are associated to sections of the vector bundle $T_GP = TP/G$ provided with the bundle coordinates $(x^\lambda, \dot{x}^\lambda, \xi^r)$ with respect to the fiber bases $\{\partial_\lambda, e_r\}$ for T_GP. The form an algebra. Given sections

$$u = u^\lambda\partial_\lambda + u^r e_r, \qquad \upsilon = \upsilon^\lambda\partial_\lambda + \upsilon^r e_r, \tag{4.21}$$

of $T_GP\to X$, their bracket reads

$$[u,\upsilon] = (u^\mu\partial_\mu\upsilon^\lambda - \upsilon^\mu\partial_\mu u^\lambda)\partial_\lambda + (u^\lambda\partial_\lambda\upsilon^r - \upsilon^\lambda\partial_\lambda u^r + c^r_{pq}u^p\upsilon^q)e_r.$$

Any section u (4.21) of the vector bundle $T_GP\to X$ yields the vector field

$$u_C = u^\lambda\partial_\lambda + (c^r_{pq}a^p_\lambda u^q + \partial_\lambda u^r - a^r_\mu\partial_\lambda u^\mu)\partial^\lambda_r$$

on the bundle of principal connections C, i.e., the infinitesimal gauge transformation with the parameter functions u^λ and u^r. Taking its vertical part and replacing parameter functions with the corresponding ghosts c^λ and c^r, we obtain the total gauge operator

$$\upsilon_e = (c^r_{pq}a^p_\lambda c^q + c^r_\lambda - a^r_\mu c^\mu_\lambda - c^\mu a^r_{\mu\lambda})\frac{\partial}{\partial a^r_\lambda}.$$

Its nilpotent extension is the BRST operator

$$\upsilon^0 = \upsilon_e + (-\frac{1}{2}c^r_{pq}c^p c^q - c^\mu c^r_\mu)\frac{\partial}{\partial c^r} + c^\lambda_\mu c^\mu \frac{\partial}{\partial c^\lambda}.$$

5 The master equation

The BGDA $P^*_\infty\{N\}$ (4.2) with the local basis (4.3) exemplifies Lagrangian systems of the following particular type.

Let $Y_0 \to X$ be a fiber bundle admitting the vertical splitting $VY_0 = Y_0 \times_X W$, where $W \to X$ is a vector bundle whose density-dual is denoted by $\overline{Y}^*_0$. Let $Y_1 \to X$ be a vector bundle and $\overline{Y}^*_1$ its density-dual. We consider the BGDA $\mathcal{P}^*_\infty[\overline{Y}^*_0; Y_1; Y_0; \overline{Y}^*_1]$ endowed with the local basis $\{y^a, \overline{y}_a\}$, where $[\overline{y}_a] = ([y^a]+1)\mathrm{mod}\, 2$. Let us call y^a and $\overline{y}_a$ the fields and antifields, respectively. Then one can associate to any graded Lagrangian

$$\mathfrak{L}\omega \in \mathcal{P}^{0,n}_\infty[\overline{Y}^*_0; Y_1; Y_0; \overline{Y}^*_1] \tag{5.1}$$

the odd graded derivations

$$\upsilon = \overleftarrow{\mathcal{E}}{}^a \partial_a = \frac{\overleftarrow{\delta}\,\mathfrak{L}}{\delta\overline{y}_a}\frac{\partial}{\partial y^a}, \qquad \overline{\upsilon} = \overleftarrow{\partial}{}^a \mathcal{E}_a = \frac{\overleftarrow{\partial}}{\partial\overline{y}_a}\frac{\delta\mathfrak{L}}{\delta y^a} \tag{5.2}$$

of the BGDA $\mathcal{P}^*_\infty[\overline{Y}^*_0; Y_1; Y_0; \overline{Y}^*_1]$.

Proposition 5.1. *The following conditions are equivalent:*

(i) the graded derivation υ (5.2) is a variational supersymmetry of a Lagrangian $\mathfrak{L}\omega$ (5.1)

(ii) the graded derivation $\overline{\upsilon}$ (5.2) is a variational supersymmetry of $\mathfrak{L}\omega$ (5.1),

(iii) the composition $(\upsilon - \overline{\upsilon})\circ(\upsilon + \overline{\upsilon})$ acting on even graded functions $f \in \mathcal{P}^0_\infty[\overline{Y}^*_0; Y_1; Y_0; \overline{Y}^*_1]$ (or, equivalently, $(\upsilon + \overline{\upsilon})\circ(\upsilon - \overline{\upsilon})$ acting on the odd ones) vanishes.

Proof. By virtue of the first variational formula (2.20) (see also Remark 2.6), the conditions (i) and (ii) are equivalent to the equality

$$\overleftarrow{\mathcal{E}}{}^a \mathcal{E}_a \omega = \frac{\overleftarrow{\delta}\,\mathfrak{L}}{\delta\overline{y}_a}\frac{\delta\mathfrak{L}}{\delta y^a}\omega = d_H\sigma. \tag{5.3}$$

In accordance with Theorem 2.2 and Proposition 3.1, the equality (5.3) is equivalent to the condition that the odd graded density $\overleftarrow{\mathcal{E}}{}^a \mathcal{E}_a \omega$ is variationally trivial. For convenience, let us replace the right variational derivatives $\overleftarrow{\mathcal{E}}{}^a$ in the equality (5.3) with the left ones $(-1)^{[a]+1}\mathcal{E}^a$. We obtain

$$\sum_a (-1)^{[a]}\mathcal{E}^a \mathcal{E}_a \omega = d_H\sigma. \tag{5.4}$$

The variational operator acting on this relation leads to the equalities

$$\sum_{0\leq|\Lambda|}(-1)^{[a]+|\Lambda|}d_\Lambda(\partial_b^\Lambda(\mathcal{E}^a\mathcal{E}_a)) = \sum_{0\leq|\Lambda|}(-1)^{[a]}[\eta(\partial_b\mathcal{E}^a)^\Lambda\mathcal{E}_{\Lambda a} + \eta(\partial_b\mathcal{E}_a)^\Lambda\mathcal{E}^a_\Lambda)] = 0,$$
$$\sum_{0\leq|\Lambda|}(-1)^{[a]+|\Lambda|}d_\Lambda(\partial^{\Lambda b}(\mathcal{E}^a\mathcal{E}_a)) = \sum_{0\leq|\Lambda|}(-1)^{[a]}[\eta(\partial^b\mathcal{E}^a)^\Lambda\mathcal{E}_{\Lambda a} + \eta(\partial^b\mathcal{E}_a)\mathcal{E}^a_\Lambda] = 0.$$

Due to the formulas (2.16), these equalities are brought into the form

$$\sum_{0\leq|\Lambda|}(-1)^{[a]}[(-1)^{[b]([a]+1)}\partial^{\Lambda a}\mathcal{E}_b\mathcal{E}_{\Lambda a} + (-1)^{[b][a]}\partial_a^\Lambda\mathcal{E}_b\mathcal{E}^a_\Lambda] = 0, \tag{5.5}$$
$$\sum_{0\leq|\Lambda|}(-1)^{[a]}[(-1)^{([b]+1)([a]+1)}\partial^{\Lambda a}\mathcal{E}^b\mathcal{E}_{\Lambda a} + (-1)^{([b]+1)[a]}\partial_a^\Lambda\mathcal{E}^b\mathcal{E}^a_\Lambda] = 0, \tag{5.6}$$

for all $\mathcal{E}_b$ and $\mathcal{E}^b$. Returning to the right variational derivatives, we obtain the relations

$$\overleftarrow{\partial}^{\Lambda a}(\mathcal{E}_b)\mathcal{E}_{\Lambda a} + (-1)^{[b]}\overleftarrow{\mathcal{E}}{}^a_\Lambda\partial_a^\Lambda\mathcal{E}_b = 0, \tag{5.7}$$
$$\overleftarrow{\mathcal{E}}{}^a_\Lambda\partial_a^\Lambda\overleftarrow{\mathcal{E}}{}^b + (-1)^{[b]+1}\overleftarrow{\partial}^{\Lambda a}(\overleftarrow{\mathcal{E}}{}^b)\mathcal{E}_{\Lambda a} = 0. \tag{5.8}$$

A direct computation shows that they are equivalent to the condition (iii). □

Following the terminology of BV quantization, we say that a graded Lagrangian (5.1) obeys the master equation (5.3).

For instance, any variationally trivial Lagrangian $L_0 \in \mathcal{P}^{0,n}_\infty[\overline{Y}{}^*_0; Y_1; Y_0; \overline{Y}{}^*_1]$ in Corollary 2.3 satisfies the master equation. We say that a solution of the master equation is not trivial if both the graded derivations (5.2) are not zero. It is readily observed that, if a graded Lagrangian $\mathfrak{L}\omega$ provides a nontrivial solution of the master equation and L_0 is a variationally trivial Lagrangian, its sum $\mathfrak{L}\omega + L_0$ is also a nontrivial solution of the master equation.

Remark 5.1. By virtue of Proposition 5.1, the master equation (5.3) is equivalent to the equalities (5.7) – (5.8). It is readily observed that these equalities are Noether identities of a Lagrangian (5.1) indexed by the variational derivatives $\mathcal{E}_b$ and $\overleftarrow{\mathcal{E}}{}^b$. Rewritten with respect to the left variational derivatives, these Noether identities take the form (5.5) – (5.6). By virtue of Theorem 4.1, the Noether identities (5.5) – (5.6) define the gauge supersymmetry u (4.11) of the Lagrangian (5.1) which is parameterized by the corresponding ghosts c_b, c^b. Using the formulas (2.16), one obtains

$$u = \sum_a\sum_{0\leq|\Lambda|}(-1)^{[a]}[c^b_\Lambda(\partial_b^\Lambda\mathcal{E}^a\partial_a + \partial_b^\Lambda\mathcal{E}_a\partial^a) + c_{\Lambda b}(\partial^{\Lambda b}\mathcal{E}^a\partial_a + \partial^{\Lambda b}\mathcal{E}_a\partial^a)],$$
$$(u^a\mathcal{E}_a + u_a\mathcal{E}^a)\omega = \sum_a\sum_{0\leq|\Lambda|}(-1)^{[a]}[c^b_\Lambda(\partial_b^\Lambda\mathcal{E}^a\mathcal{E}_a + \partial_b^\Lambda\mathcal{E}_a\mathcal{E}^a) + c_{\Lambda b}(\partial^{\Lambda b}\mathcal{E}^a\mathcal{E}_a + \partial^{\Lambda b}\mathcal{E}_a\mathcal{E}^a)]\omega =$$
$$\sum_{0\leq|\Lambda|}(c^b_\Lambda\partial^\Lambda_b + c_{\Lambda b}\partial^{\Lambda b})(\sum_a(-1)^{[a]}\mathcal{E}^a\mathcal{E}_a\omega) = d_H[\sum_{0\leq|\Lambda|}(c^b_\Lambda\partial^\Lambda_b + c_{\Lambda b}\partial^{\Lambda b})\sigma],$$

where the last equality results from action of the graded derivation $c_b\partial^b + c^b\partial_b$ on the both sides of the master equation (5.4).

Let us return to an original Lagrangian system $(\mathcal{S}^*_\infty[F;Y], L)$ and its extension $(P^*_\infty\{N\}, L_e)$ to ghosts and antifields, together with the odd graded derivations (5.2) which read

$$\upsilon_e = \vartheta + \vartheta_e = \frac{\overleftarrow{\delta}\mathcal{L}_1}{\delta\overline{s}_A}\frac{\partial}{\partial s^A} + \sum_{0\leq k\leq N}\frac{\overleftarrow{\delta}\mathcal{L}_1}{\delta\overline{c}_{r_k}}\frac{\partial}{\partial c^{r_k}}, \tag{5.9}$$

$$\overline{\upsilon}_e = \overline{\vartheta} + \delta_N = \frac{\overleftarrow{\partial}}{\partial\overline{s}_A}\frac{\delta\mathcal{L}_1}{\delta s^A} + [\frac{\overleftarrow{\partial}}{\partial\overline{s}_A}\frac{\delta\mathcal{L}}{\delta s^A} + \sum_{0\leq k\leq N}\frac{\overleftarrow{\partial}}{\partial\overline{c}_{r_k}}\frac{\delta\mathcal{L}_1}{\delta c^{r_k}}]. \tag{5.10}$$

An original Lagrangian provides a trivial solution of the master equation. It follows at once from the equality (4.7) that the graded Lagrangian L_e (4.6) satisfies the master equation (5.3) iff

$$\frac{\overleftarrow{\delta}\mathcal{L}_1}{\delta\overline{s}_A}\frac{\delta\mathcal{L}_1}{\delta s^A}\omega = d_H\sigma'. \tag{5.11}$$

If the condition (5.11) does not hold, a problem is to extend the graded Lagrangian L_e (4.6) to a solution of the master equation

$$L_e + L' = L + L_1 + L_2 + \cdots \tag{5.12}$$

by means of terms L_i of polynomial degree $i > 1$ in ghosts. They are assumed to be even of zero antifield number. Such an extension need not exists. Our goal is to investigate the conditions of its existence (Theorems 5.2 and 5.3).

Let a graded Lagrangian (5.12) be a solution of the master equation (5.3), which reads

$$\overleftarrow{\delta}^A(L_1+L')\delta_A(L+L_1+L') + \sum_{0\leq k\leq N}\overleftarrow{\delta}^{r_k}(L_1+L')\delta_{r_k}(L_1+L') = d_H\sigma. \tag{5.13}$$

As was mentioned above, such a solution is never unique, but it is defined at least up to a d_H-exact density. The master equation (5.13) is decomposed into a set of equalities with respect to the polynomial degree in ghosts. We have

$$\overleftarrow{\delta}^A(L_1)\delta_A L + \sum_{0\leq k\leq N}\overleftarrow{\delta}^{r_k}(L_1)\delta_{r_k}L_1 = d_H\sigma_1, \tag{5.14}$$

$$\begin{aligned}&\sum_{1\leq j<i}\overleftarrow{\delta}^A(L_j)\delta_A L_{i-j} + \sum_{1<j<i}\sum_{0\leq k\leq N}\overleftarrow{\delta}^{r_k}(L_j)\delta_{r_k}L_{i-j+1} + \\ &\qquad \overleftarrow{\delta}^A(L_i)\delta_A L + \sum_{0\leq k\leq N}[\overleftarrow{\delta}^{r_k}(L_1)\delta_{r_k}L_i + \overleftarrow{\delta}^{r_k}(L_i)\delta_{r_k}L_1] = d_H\sigma_i, \quad i\geq 2.\end{aligned} \tag{5.15}$$

The first one is exactly the equality (4.7). The others are brought into the form

$$\sum_{1\leq j<i}\overleftarrow{\delta}^A(L_j)\delta_A L_{i-j} + \sum_{1<j<i}\sum_{0\leq k\leq N}\overleftarrow{\delta}^{r_k}(L_j)\delta_{r_k}L_{i-j+1} + \gamma(L_i) = d_H\sigma'_i, \quad i\geq 2, \tag{5.16}$$

$$\gamma = \delta_N + \vartheta_e = \frac{\overleftarrow{\partial}}{\partial\overline{s}_A}\frac{\delta\mathcal{L}}{\delta s^A} + \sum_{0\leq k\leq N}[\frac{\overleftarrow{\partial}}{\partial\overline{c}_{r_k}}\frac{\delta\mathcal{L}_1}{\delta c^{r_k}} + \frac{\overleftarrow{\delta}\mathcal{L}_1}{\delta\overline{c}_{r_k}}\frac{\partial}{\partial c^{r_k}}]. \tag{5.17}$$

It is readily observed that a graded Lagrangian (5.12) obeys the master equation (5.13) iff its term L_2 is a solution of the equation (5.16), $i = 2$, the term L_3 satisfies the equation (5.16),

$i = 3$, and so on. Since γ (5.17) vanishes on functions $f \in \mathcal{S}^0_\infty[F; Y]$, each equation (5.16) reduces to a system of linear algebraic equations with coefficients in the ring $\mathcal{S}^0_\infty[F; Y]$ whose homogeneous part is given by the operator γ (5.17). Because this operator is not invertible as a rule, a solution of the equations (5.16) need not exist. In order to study its existence, let us consider the condition (iii) in Proposition 5.1.

Theorem 5.2. *The graded Lagrangian L_e (4.6) can be extended to a solution (5.12) of the master equation only if the graded derivation u_e (4.5) is extended to a graded derivation nilpotent on the shell.*

Proof. Given a graded Lagrangian (5.12), the corresponding graded derivations (5.2) read

$$\upsilon = \frac{\overleftarrow{\delta}(\mathcal{L}_1 + \mathcal{L}')}{\delta \overline{s}_A}\frac{\partial}{\partial s^A} + \sum_{0 \leq k \leq N} \frac{\overleftarrow{\delta}(\mathcal{L}_1 + \mathcal{L}')}{\delta \overline{c}_{r_k}}\frac{\partial}{\partial c^{r_k}}, \tag{5.18}$$

$$\overline{\upsilon} = \frac{\overleftarrow{\partial}}{\partial \overline{s}_A}\frac{\delta(\mathcal{L} + \mathcal{L}_1 + \mathcal{L}')}{\delta s^A} + \sum_{0 \leq k \leq N} \frac{\overleftarrow{\partial}}{\partial \overline{c}_{r_k}}\frac{\delta(\mathcal{L}_1 + \mathcal{L}')}{\delta c^{r_k}}. \tag{5.19}$$

Then the condition (iii) can be written in the form (2.22) as

$$(\upsilon + \overline{\upsilon})(\upsilon) = 0, \qquad (\upsilon + \overline{\upsilon})(\overline{\upsilon}) = 0. \tag{5.20}$$

It falls into a set of equalities with respect to the polynomial degree in antifields. Let us put

$$\upsilon = \upsilon^0 + \upsilon^1 + \upsilon', \qquad \overline{\upsilon} = \overline{\upsilon}^0 + \overline{\upsilon}',$$

where υ^0 and υ^1 are the parts of υ (5.18) of zero and first polynomial degree in antifields, respectively, and $\overline{\upsilon}^0$ is that of $\overline{\upsilon}$ (5.19) independent of antifields. It is readily observed that $\overline{\upsilon}^0 = \overline{\delta}$ is the Koszul–Tate differential. Let us consider the part of the equalities (5.20) which is independent of antifields. It reads

$$\upsilon^0(\upsilon^0) + \overline{\upsilon}^0(\upsilon^1) = \upsilon^0(\upsilon^0) + \overline{\delta}(\upsilon^1) = 0, \tag{5.21}$$

i.e., the graded derivation υ^0 vanishes on the shell. It is readily observed that the part of υ^0 linear in ghosts is exactly the total gauge operator u_e (4.5), i.e., υ^0 provides a nilpotent extension u_e on the shell. □

In other words, the Lagrangian L_e (4.6) is extended to a solution of the master equation only if the gauge and higher-stage gauge supersymmetries of an original Lagrangian L form an algebra on the shell.

In order to formulate the sufficient condition, let us assume that the gauge and higher-stage gauge supersymmetries of an original Lagrangian L form an algebra, and this algebra is given from the beginning, i.e., we have a nilpotent extension υ^0 of the total gauge operator u_e (4.5).

Theorem 5.3. *Let the graded derivation u_e (4.5) admit a nilpotent extension υ^0 (4.16) of zero antifield number. Then the graded Lagrangian*

$$\mathfrak{L}\omega = L_e + \sum_{1 \leq k \leq N} \xi^{r_{k-1}} \overline{c}_{r_{k-1}} \omega \tag{5.22}$$

satisfies the master equation.

Proof. If the graded derivation υ^0 is nilpotent, then $\overline{\delta}(\upsilon^1) = 0$ by virtue of the equation (5.21). It follows that the part L_1^2 of the Lagrangian L_e quadratic in antifields obeys the relations $\overline{\delta}(\overleftarrow{\delta}^{r_k}(\mathcal{L}_1^2)) = 0$ for all indices r_k. This part consists of the terms $h_{r_k}^{(r_{k-2},\Sigma)(A,\Xi)}\overline{c}_{\Sigma r_{k-2}}\overline{s}_{\Xi A}$ (3.23), which consequently are $\overline{\delta}$-closed. Then the summand G_{r_k} of each cocycle Δ_{r_k} (3.23) is δ_{k-1}-closed in accordance with the relation (3.32). It follows that its summand h_{r_k} is also δ_{k-1}-closed and, consequently, δ_{k-2}-closed. Hence it is δ_{k-1}-exact by virtue the homology regularity condition. Therefore, Δ_{r_k} is reduced only to the summand G_{r_k} linear in antifields. It follows that the Lagrangian L_1 (4.6) is linear in antifields. In this case, we have

$$u^A = \overleftarrow{\delta}^A(\mathcal{L}_e), \qquad u^{r_k} = \overleftarrow{\delta}^{r_k}(\mathcal{L}_e)$$

for all indices A and r_k and, consequently,

$$\upsilon^A = \overleftarrow{\delta}^A(\mathfrak{L}), \qquad \upsilon^{r_k} = \overleftarrow{\delta}^{r_k}(\mathfrak{L}),$$

i.e., υ^0 is the graded derivation υ (5.2) defined by the Lagrangian (5.22). Then the nilpotency condition $\upsilon^0(\upsilon^0) = 0$ takes the form

$$\upsilon^0(\overleftarrow{\delta}^A(\mathfrak{L})) = 0, \qquad \upsilon^0(\overleftarrow{\delta}^{r_k}(\mathfrak{L})) = 0.$$

Hence, we obtain

$$\upsilon^0(\mathfrak{L}) = \upsilon^0(\overleftarrow{\delta}^A(\mathfrak{L})\overline{s}_A + \overleftarrow{\delta}^{r_k}(\mathfrak{L})\overline{c}_{r_k}) = 0,$$

i.e., υ^0 is a variational supersymmetry of the Lagrangian (5.22). Thus, it satisfies the master equation in accordance with Proposition 5.1 □

Since the summand L_1 of the Lagrangian (5.22) is linear in antifields, the Lagrangian (5.22) up to a d_H-exact term can be written in the form

$$\mathfrak{L}\omega = L + u^A\overline{s}_A + \sum_{1\leq k\leq N}(u^{r_{k-1}} + \xi^{r_{k-1}})\overline{c}_{r_{k-1}}\omega, \tag{5.23}$$

which is also a solution of the master equation.

6 Example

We address the topological BF theory of two exterior forms A and B of form degree $|A|+|B| = \dim X - 1$ on a smooth manifold X [9]. It is a reducible degenerate Lagrangian theory [4]. Since the verification of the homology regularity condition in a general case is rather complicated, we here restrict our consideration to the simplest example of the topological BF theory when A is a function [6].

Let us consider the fiber bundle

$$Y = \mathbb{R} \underset{X}{\times} \overset{n-1}{\wedge} T^*X,$$

coordinated by $(x^\lambda, A, B_{\mu_1\dots\mu_{n-1}})$ and provided with the canonical $(n-1)$-form

$$B = \frac{1}{(n-1)!} B_{\mu_1\dots\mu_{n-1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{n-1}}.$$

The Lagrangian and the Euler–Lagrange operator of the topological BF theory in question read

$$L_{\rm BF} = \frac{1}{n} A d_H B, \tag{6.1}$$

$$\delta L = dA \wedge \mathcal{E}\omega + dB_{\mu_1\dots\mu_{n-1}} \wedge \mathcal{E}^{\mu_1\dots\mu_{n-1}}\omega,$$

$$\mathcal{E} = \epsilon^{\mu\mu_1\dots\mu_{n-1}} d_\mu B_{\mu_1\dots\mu_{n-1}}, \qquad \mathcal{E}^{\mu_1\dots\mu_{n-1}} = -\epsilon^{\mu\mu_1\dots\mu_{n-1}} d_\mu A, \tag{6.2}$$

where ϵ is the Levi–Civita symbol.

Let us extend the BGDA $\mathcal{O}^*_\infty Y$ to the BGDA $\mathcal{P}^*_\infty[\overline{Y}^*; Y]$ where

$$VY = Y \underset{X}{\times} Y, \qquad \overline{Y}^* = (\mathbb{R} \underset{X}{\times} \overset{n-1}{\wedge} TX) \underset{X}{\otimes} \overset{n}{\wedge} T^*X.$$

This BGDA possesses the local basis $\{A, B_{\mu_1\dots\mu_{n-1}}, \overline{s}, \overline{s}^{\mu_1\dots\mu_{n-1}}\}$, where $\overline{s}, \overline{s}^{\mu_1\dots\mu_{n-1}}$ are odd antifields of antifield number 1. With the nilpotent Koszul–Tate differential

$$\overline{\delta} = \frac{\overleftarrow{\partial}}{\partial \overline{s}} \mathcal{E} + \frac{\overleftarrow{\partial}}{\partial \overline{s}^{\mu_1\dots\mu_{n-1}}} \mathcal{E}^{\mu_1\dots\mu_{n-1}},$$

we have the complex (3.3),

$$0 \leftarrow \mathrm{Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{Y}^*; Y]_1 \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{Y}^*; Y]_2.$$

A generic one-chain reads

$$\Phi = \sum_{0\leq|\Lambda|} (\Phi^\Lambda \overline{s}_\Lambda + \Phi^\Lambda_{\mu_1\dots\mu_{n-1}} \overline{s}^{\mu_1\dots\mu_{n-1}}_\Lambda)\omega,$$

and the cycle condition $\overline{\delta}\Phi = 0$ takes the form

$$\Phi^\Lambda \mathcal{E}_\Lambda + \Phi^\Lambda_{\mu_1\dots\mu_{n-1}} \mathcal{E}^{\mu_1\dots\mu_{n-1}}_\Lambda = 0. \tag{6.3}$$

If Φ^Λ and $\Phi^\Lambda_{\mu_1\dots\mu_{n-1}}$ are independent of the variational derivatives (6.2) (i.e., Φ is a nontrivial cycle), the equality (6.3) is split into the following:

$$\Phi^\Lambda \mathcal{E}_\Lambda = 0, \qquad \Phi^\Lambda_{\mu_1\dots\mu_{n-1}} \mathcal{E}^{\mu_1\dots\mu_{n-1}}_\Lambda = 0.$$

The first equality holds iff $\Phi^\Lambda = 0$, i.e., there is no Noether identity involving $\mathcal{E}$. The second one is satisfied iff

$$\Phi^{\lambda_1\dots\lambda_k}_{\mu_1\dots\mu_{n-1}} \epsilon^{\mu\mu_1\dots\mu_{n-1}} = -\Phi^{\mu\lambda_2\dots\lambda_k}_{\mu_1\dots\mu_{n-1}} \epsilon^{\lambda_1\mu_1\dots\mu_{n-1}}.$$

It follows that Φ factorizes as

$$\Phi = \sum_{0\leq|\Xi|} G^\Xi_{\nu_2\dots\nu_{n-1}} d_\Xi \Delta^{\nu_2\dots\nu_{n-1}}\omega$$

via local graded densities

$$\Delta^{\nu_2\ldots\nu_{n-1}} = \Delta^{\nu_2\ldots\nu_{n-1},\lambda}_{\alpha_1\ldots\alpha_{n-1}}\overline{s}^{\alpha_1\ldots\alpha_{n-1}}_\lambda = \delta^\lambda_{\alpha_1}\delta^{\nu_2}_{\alpha_2}\cdots\delta^{\nu_{n-1}}_{\alpha_{n-1}}\overline{s}^{\alpha_1\ldots\alpha_{n-1}}_\lambda = d_{\nu_1}\overline{s}^{\nu_1\nu_2\ldots\nu_{n-1}}, \qquad (6.4)$$

which provide the complete Noether identities

$$d_{\nu_1}\mathcal{E}^{\nu_1\nu_2\ldots\nu_{n-1}} = 0. \qquad (6.5)$$

The local graded densities (6.4) form the basis for a projective $C^\infty(X)$-module of finite rank which is isomorphic to the module of sections of the vector bundle

$$\overline{V}^* = \overset{n-2}{\wedge} TX \underset{X}{\otimes} \overset{n}{\wedge} T^*X, \qquad V = \overset{n-2}{\wedge} T^*X.$$

Therefore, let us extend the BGDA $\mathcal{P}^*_\infty[\overline{Y}^*;Y]$ to the BGDA $\mathcal{P}^*_\infty\{0\} = \mathcal{P}^*_\infty[\overline{Y}^*;Y;V]$ possessing the local basis $\{A, B_{\mu_1\ldots\mu_{n-1}}, \overline{s}, \overline{s}^{\mu_1\ldots\mu_{n-1}}, \overline{c}^{\mu_2\ldots\mu_{n-1}}\}$, where $\overline{c}^{\mu_2\ldots\mu_{n-1}}$ are even antifields of antifield number 2. We have the nilpotent graded derivation

$$\delta_0 = \overline{\delta} + \frac{\overleftarrow{\partial}}{\partial\overline{c}^{\mu_2\ldots\mu_{n-1}}}\Delta^{\mu_2\ldots\mu_{n-1}}$$

of $\mathcal{P}^*_\infty\{0\}$. Its nilpotency is equivalent to the Noether identities (6.5). Then we obtain the one-exact complex

$$0 \leftarrow \mathrm{Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{Y}^*;Y]_1 \xleftarrow{\delta_0} \mathcal{P}^{0,n}_\infty\{0\}_2 \xleftarrow{\delta_0} \mathcal{P}^{0,n}_\infty\{0\}_3.$$

Iterating the arguments, we come to the $(N+1)$-exact complex (3.24) for $N \leq n-3$ as follows. Let us consider the vector bundles

$$V_k = \overset{n-k-2}{\wedge} T^*X, \qquad k = 1,\ldots,N,$$

and the corresponding BGDA $\mathcal{P}^*_\infty\{N\} = \mathcal{P}^*_\infty[...V_3V_1\overline{Y}^*;Y;VV_2V_4...]$, possessing the local basis

$$\{A, B_{\mu_1\ldots\mu_{n-1}}, \overline{s}, \overline{s}^{\mu_1\ldots\mu_{n-1}}, \overline{c}^{\mu_2\ldots\mu_{n-1}}, \ldots, \overline{c}^{\mu_{N+2}\ldots\mu_{n-1}}\},$$
$$[\overline{c}^{\mu_{k+2}\ldots\mu_{n-1}}] = (k+1)\mathrm{mod}\,2, \qquad \mathrm{Ant}[\overline{c}^{\mu_{k+2}\ldots\mu_{n-1}}] = k+3.$$

It is provided with the nilpotent graded derivation

$$\delta_N = \delta_0 + \sum_{1\leq k\leq N}\frac{\overleftarrow{\partial}}{\partial\overline{c}^{\mu_{k+2}\ldots\mu_{n-1}}}\Delta^{\mu_{k+2}\ldots\mu_{n-1}}, \qquad \Delta^{\mu_{k+2}\ldots\mu_{n-1}} = d_{\mu_{k+1}}\overline{c}^{\mu_{k+1}\mu_{k+2}\ldots\mu_{n-1}}, \qquad (6.6)$$

of antifield number -1. Its nilpotency results from the Noether identities (6.5) and the equalities

$$d_{\mu_{k+2}}\Delta^{\mu_{k+2}\ldots\mu_{n-1}} = 0, \qquad k = 0,\ldots,N, \qquad (6.7)$$

which are k-stage Noether identities [4]. Then the manifested $(N+1)$-exact complex reads

$$0 \leftarrow \mathrm{Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{Y}^*;Y]_1 \xleftarrow{\delta_0} \mathcal{P}^{0,n}_\infty\{0\}_2 \xleftarrow{\delta_1} \mathcal{P}^{0,n}_\infty\{1\}_3 \cdots \qquad (6.8)$$
$$\xleftarrow{\delta_{N-1}} \mathcal{P}^{0,n}_\infty\{N-1\}_{N+1} \xleftarrow{\delta_N} \mathcal{P}^{0,n}_\infty\{N\}_{N+2} \xleftarrow{\delta_N} \mathcal{P}^{0,n}_\infty\{N\}_{N+3}.$$

It obeys the following $(N+2)$-homology regularity condition (see Appendix B for the proof).

Lemma 6.1. *Any $(N+2)$-cycle $\Phi \in \mathcal{P}^{0,n}_\infty\{N-1\}_{N+2}$ up to a δ_{N-1}-boundary takes the form*

$$\Phi = \sum_{k_1+\cdots+k_i+3i=N+2} \sum_{0\leq|\Lambda_1|,\ldots,|\Lambda_i|} G^{\Lambda_1\cdots\Lambda_i}_{\mu^1_{k_1+2}\cdots\mu^1_{n-1};\ldots;\mu^i_{k_i+2}\cdots\mu^i_{n-1}} \tag{6.9}$$
$$d_{\Lambda_1}\Delta^{\mu^1_{k_1+2}\cdots\mu^1_{n-1}}\cdots d_{\Lambda_i}\Delta^{\mu^i_{k_i+2}\cdots\mu^i_{n-1}}\omega, \qquad k=-1,0,1,\ldots,N,$$

where $k=-1$ stands for $\overline{c}^{\mu_1\cdots\mu_{n-1}}=\overline{s}^{\mu_1\cdots\mu_{n-1}}$ and $\Delta^{\mu_1\cdots\mu_{n-1}}=\mathcal{E}^{\mu_1\cdots\mu_{n-1}}$. It follows that Φ is a δ_N-boundary.

Following the proof of Lemma 6.1, one can also show that any $(N+2)$-cycle $\Phi \in \mathcal{P}^{0,n}_\infty\{N\}_{N+2}$ up to a boundary takes the form

$$\Phi = \sum_{0\leq|\Lambda|} G^{\Lambda}_{\mu_{N+2}\cdots\mu_{n-1}} d_\Lambda \Delta^{\mu_{N+2}\cdots\mu_{n-1}}\omega,$$

i.e., the homology $H_{N+2}(\delta_N)$ of the complex (6.8) is finitely generated by the cycles $\Delta^{\mu_{N+2}\cdots\mu_{n-1}}$. Thus, the complex (6.8) admits the $(N+2)$-exact extension (3.31).

The iteration procedure is prolonged till $N=n-3$. We have the BGDA $\mathcal{P}^*\{n-2\}$, where $V_{n-2}=X\times\mathbb{R}$. It possesses the local basis

$$\{A, B_{\mu_1\ldots\mu_{n-1}}, \overline{s}, \overline{s}^{\mu_1\cdots\mu_{n-1}}, \overline{c}^{\mu_2\cdots\mu_{n-1}}, \ldots, \overline{c}^{\mu_{n-1}}, \overline{c}\},$$

where $[\overline{c}]=(n-1)\mathrm{mod}\,2$ and $\mathrm{Ant}[\overline{c}]=n+1$. The corresponding Koszul–Tate complex reads

$$0 \leftarrow \mathrm{Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{Y}^*;Y]_1 \xleftarrow{\delta_0} \mathcal{P}^{0,n}_\infty\{0\}_2 \xleftarrow{\delta_1} \mathcal{P}^{0,n}_\infty\{1\}_3\cdots$$
$$\xleftarrow{\delta_{n-3}} \mathcal{P}^{0,n}_\infty\{n-3\}_{n-1} \xleftarrow{\delta_{n-2}} \mathcal{P}^{0,n}_\infty\{n-2\}_n \xleftarrow{\delta_{n-2}} \mathcal{P}^{0,n}_\infty\{n-2\}_{n+1}.$$
$$\delta_{n-2}=\delta_0+\sum_{1\leq k\leq n-3}\frac{\overleftarrow{\partial}}{\partial\overline{c}^{\mu_{k+2}\cdots\mu_{n-1}}}\Delta^{\mu_{k+2}\cdots\mu_{n-1}}+\frac{\overleftarrow{\partial}}{\partial\overline{c}}\Delta, \qquad \Delta=d_{\mu_{n-1}}\overline{c}^{\mu_{n-1}}.$$

Let us enlarge the BGDA $\mathcal{P}^*\{n-2\}$ to the BGDA $P^*\{n-2\}$ (4.2) possessing the local basis

$$\{A, B_{\mu_1\ldots\mu_{n-1}}, c_{\mu_2\ldots\mu_{n-1}},\ldots,c_{\mu_{n-1}}, c, \overline{s}, \overline{s}^{\mu_1\cdots\mu_{n-1}}, \overline{c}^{\mu_2\cdots\mu_{n-1}},\ldots,\overline{c}^{\mu_{n-1}},\overline{c}\},$$

where $c_{\mu_2\ldots\mu_{n-1}},\ldots,c_{\mu_{n-1}},c$ are the corresponding ghosts. Let us extend the Lagrangian L_{BF} (6.1) to the even graded density

$$L_e = L_{\mathrm{BF}}+L_1 = L_{\mathrm{BF}}+[\sum_{0\leq k\leq n-3} c_{\mu_{k+2}\cdots\mu_{n-1}}\Delta^{\mu_{k+2}\cdots\mu_{n-1}}+c\Delta]\omega = \tag{6.10}$$
$$L_{\mathrm{BF}}+[c_{\mu_2\ldots\mu_{n-1}}d_{\mu_1}\overline{s}^{\mu_1\mu_2\cdots\mu_{n-1}}+\sum_{1\leq k\leq n-3}c_{\mu_{k+2}\cdots\mu_{n-1}}d_{\mu_{k+1}}\overline{c}^{\mu_{k+1}\mu_{k+2}\cdots\mu_{n-1}}+cd_{\mu_{n-1}}\overline{c}^{\mu_{n-1}}]\omega.$$

Since the graded density L_1 is independent on A and $B_{\mu_1\ldots\mu_{n-1}}$, the relation (5.11) holds and, therefore, L_e (6.10) is a solution of the master equation.

7 Appendix A

We start the proof of Theorem 2.2 with the following algebraic Poincaré lemma.

Lemma 7.1. *If $Y = \mathbb{R}^{n+m} \to \mathbb{R}^n$, the complex (2.12) at all the terms, except $\mathbb{R}$, is exact, while the complex (2.13) is exact.*

Proof. This is the case of an affine bundle Y, and the above mentioned exactness has been proved when the ring $\mathcal{O}^0_\infty Y$ is restricted to the subring $\mathcal{P}^0_\infty Y$ of polynomial functions (see [18], Lemmas 4.2 – 4.3). The proof of these lemmas is straightforwardly extended to $\mathcal{O}^0_\infty Y$ if the homotopy operator (4.5) in [18], Lemma 4.2 is replaced with that (4.8) in [18], Remark 4.1. $\Box$

The proof of Theorem 2.2 follows that of [18], Theorem 2.1. We first prove Theorem 2.2 for the above mentioned BGDA $\Gamma(\mathfrak{T}^*_\infty[F;Y])$. Similarly to $\mathcal{S}^*_\infty[F;Y]$, the sheaf $\mathfrak{T}^*_\infty[F;Y]$ and the BGDA $\Gamma(\mathfrak{T}^*_\infty[F;Y])$ are split into the variational bicomplexes, and we consider their subcomplexes

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{T}^0_\infty[F;Y] \xrightarrow{d_H} \mathfrak{T}^{0,1}_\infty[F;Y] \cdots \xrightarrow{d_H} \mathfrak{T}^{0,n}_\infty[F;Y] \xrightarrow{\delta} \mathfrak{E}_1, \tag{7.1}$$

$$0 \to \mathfrak{T}^{1,0}_\infty[F;Y] \xrightarrow{d_H} \mathfrak{T}^{1,1}_\infty[F;Y] \cdots \xrightarrow{d_H} \mathfrak{T}^{1,n}_\infty[F;Y] \xrightarrow{\varrho} \mathfrak{E}_1 \to 0, \tag{7.2}$$

$$0 \longrightarrow \mathbb{R} \longrightarrow \Gamma(\mathfrak{T}^0_\infty[F;Y]) \xrightarrow{d_H} \Gamma(\mathfrak{T}^{0,1}_\infty[F;Y]) \cdots \xrightarrow{d_H} \Gamma(\mathfrak{T}^{0,n}_\infty[F;Y]) \xrightarrow{\delta} \Gamma(\mathfrak{E}_1), \tag{7.3}$$

$$0 \to \Gamma(\mathfrak{T}^{1,0}_\infty[F;Y]) \xrightarrow{d_H} \Gamma(\mathfrak{T}^{1,1}_\infty[F;Y]) \cdots \xrightarrow{d_H} \Gamma(\mathfrak{T}^{1,n}_\infty[F;Y]) \xrightarrow{\varrho} \Gamma(\mathfrak{E}_1) \to 0, \tag{7.4}$$

where $\mathfrak{E}_1 = \varrho(\mathfrak{T}^{1,n}_\infty[F;Y])$. By virtue of Lemma 7.1, the complexes (7.1) – (7.2) at all the terms, except $\mathbb{R}$, are exact. The terms $\mathfrak{T}^{*,*}_\infty[F;Y]$ of the complexes (7.1) – (7.2) are sheaves of $\Gamma(\mathfrak{T}^0_\infty)$-modules. Since $J^\infty Y$ admits a partition of unity just by elements of $\Gamma(\mathfrak{T}^0_\infty)$, these sheaves are fine and, consequently, acyclic. By virtue of the abstract de Rham theorem (see [18], Theorem 8.4, generalizing [22], Theorem 2.12.1), cohomology of the complex (7.3) equals the cohomology of $J^\infty Y$ with coefficients in the constant sheaf $\mathbb{R}$ and, consequently, the de Rham cohomology of Y, which is the strong deformation retract of $J^\infty Y$. Similarly, the complex (7.4) is proved to be exact. It remains to prove that cohomology of the complexes (2.12) – (2.13) equals that of the complexes (7.3) – (7.4). The proof of this fact straightforwardly follows the proof of [18], Theorem 2.1, and it is a slight modification of the proof of [18], Theorem 4.1, where graded exterior forms on the infinite order jet manifold $J^\infty Y$ of an affine bundle are treated as those on X.

8 Appendix B

In order to prove Lemma 6.1, let us choose some basis element $\overline{c}^{\mu_{k+2}\cdots\mu_{n-1}}$ and denote it simply by $\overline{c}$. Let Φ contain a summand $\phi_1\overline{c}$, linear in $\overline{c}$. Then the cycle condition reads

$$\delta_{N-1}\Phi = \delta_{N-1}(\Phi - \phi_1\overline{c}) + (-1)^{[\overline{c}]}\delta_{N-1}(\phi_1)\overline{c} + \phi\Delta = 0, \qquad \Delta = \delta_{N-1}\overline{c}.$$

It follows that Φ contains a summand $\psi\Delta$ such that

$$(-1)^{[\overline{c}]+1}\delta_{N-1}(\psi)\Delta + \phi\Delta = 0.$$

This equality implies the relation

$$\phi_1 = (-1)^{[\overline{c}]+1}\delta_{N-1}(\psi) \tag{8.1}$$

because the reduction conditions (6.7) involve total derivatives of Δ, but not Δ. Hence,

$$\Phi = \Phi' + \delta_{N-1}(\psi\overline{c}),$$

where Φ' contains no term linear in $\overline{c}$. Furthermore, let $\overline{c}$ be even and Φ has a summand $\sum \phi_r \overline{c}^r$ polynomial in $\overline{c}$. Then the cycle condition leads to the equalities

$$\phi_r\Delta = -\delta_{N-1}\phi_{r-1}, \qquad r \geq 2.$$

Since ϕ_1 (8.1) is δ_{N-1}-exact, then $\phi_2 = 0$ and, consequently, $\phi_{r>2} = 0$. Thus, a cycle Φ up to a δ_{N-1}-boundary contains no term polynomial in $\overline{c}$. It reads

$$\Phi = \sum_{k_1+\cdots+k_i+3i=N+2} \sum_{0<|\Lambda_1|,\ldots,|\Lambda_i|} G^{\Lambda_1\cdots\Lambda_i}_{\mu^1_{k_1+2}\cdots\mu^1_{n-1};\ldots;\mu^i_{k_i+2}\cdots\mu^i_{n-1}} \overline{c}^{\mu^1_{k_1+2}\cdots\mu^1_{n-1}}_{\Lambda_1}\cdots\overline{c}^{\mu^i_{k_i+2}\cdots\mu^i_{n-1}}_{\Lambda_i}\omega. \tag{8.2}$$

However, the terms polynomial in $\overline{c}$ may appear under general covariant transformations

$$\overline{c}'^{\nu_{k+2}\cdots\nu_{n-1}} = \det(\frac{\partial x^\alpha}{\partial x'^\beta})\frac{\partial x'^{\nu_{k+2}}}{\partial x^{\mu_{k+2}}}\cdots\frac{\partial x'^{\nu_{n-1}}}{\partial x^{\mu_{n-1}}}\overline{c}^{\mu_{k+2}\cdots\mu_{n-1}}$$

of a chain Φ (8.2). In particular, Φ contains the summand

$$\sum_{k_1+\cdots+k_i+3i=N+2} F_{\nu^1_{k_1+2}\cdots\nu^1_{n-1};\ldots;\nu^i_{k_i+2}\cdots\nu^i_{n-1}}\overline{c}'^{\nu^1_{k_1+2}\cdots\nu^1_{n-1}}\cdots\overline{c}'^{\nu^i_{k_i+2}\cdots\nu^i_{n-1}},$$

which must vanish if Φ is a cycle. This takes place only if Φ factorizes through the graded densities $\Delta^{\mu_{k+2}\cdots\mu_{n-1}}$ (6.6) in accordance with the expression (6.9).

References

[1] Anderson, I.: Introduction to the variational bicomplex. *Contemp. Math.* **132**, 51-73 (1992)

[2] Barnich, G., Brandt, F. and Henneaux, M.: Local BRST cohomology in gauge theories. *Phys. Rep.* **338**, 439-569 (2000)

[3] Bartocci, C., Bruzzo, U. and Hernández Ruipérez, D.: *The Geometry of Supermanifolds.* Dordrecht: Kluwer, 1991

[4] Bashkirov, D., Giachetta, G., Mangiarotti, L. and Sardanashvily, G.: Noether's second theorem in a general setting. Reducible gauge theories. *J. Phys. A* **38**, 5329-5344 (2005); *E-print arXiv:* math.DG/0411070

[5] Bashkirov, D., Giachetta, G., Mangiarotti, L. and Sardanashvily, G.: Noether's second theorem for BRST symmetries. *J. Math. Phys.* **46**, 053517 (2005); *E-print arXiv:* math-ph/0412034

[6] Bashkirov, D., Giachetta, G., Mangiarotti, L. and Sardanashvily, G.: The antifield Koszul–Tate complex of reducible Noether identities. *J. Math. Phys.* **46**, 103513 (2005); *E-print arXiv:* math-ph/0506034

[7] Bashkirov, D. and Sardanashvily, G.: On the BV quantization of gauge gravitation theory. *Int. J. Geom. Methods Mod. Phys.* **2**, 203-226 (2005); *E-print arXiv:* hep-th/0501254

[8] Batalin, I. and Vilkovisky, G.: Closure of the gauge algebra, generalized Lie algebra equations and Feynman rules. *Nucl. Phys.* **B234**, 106-124 (1984)

[9] Birmingham, D., Blau, M., Rakowski, M. and Thompson, G.: Topological field theories. *Phys. Rep.* **209**, 129-340 (1991)

[10] Brandt, F.: Jet coordinates for local BRST cohomology. *Lett. Math. Phys.* **55**, 149-159 (2001)

[11] Fisch, J., Henneaux, M., Stasheff, J. and Teitelboim, C.: Existence, uniqueness and cohomology of the classical BRST charge with ghosts of ghosts. *Commun. Math. Phys.* **120**, 379-407 (1989)

[12] Fisch, J. and Henneaux, M.: Homological perturbation theory and algebraic structure of the antifield-antibracket formalism for gauge theories *Commun. Math. Phys.* **128**, 627-640 (1990)

[13] Fuks, D.: *Cohomology of Infinite-Dimensional Lie Algebras.* N.Y.: Consultants Bureau, 1986

[14] Fulp, R., Lada, T. and Stasheff, J.: Sh-Lie algebras induced by gauge transformations. *Commun.. Math. Phys.* **231**, 25-43 (2002)

[15] Fulp, R., Lada, T. and Stasheff, J.: Noether variational Theorem II and the BV formalism. *Rend. Circ. Mat. Palermo (2) Suppl.* No. 71, 115-126 (2003)

[16] Fuster, A., Henneaux, M. and Maas, A.: BRST quantization: a short review. *Int. J. Geom. Methods Mod. Phys.* **2**, 939-964 (2005); *E-print arXiv*: hep-th/0506098

[17] Giachetta, G., Mangiarotti, L. and Sardanashvily, G.: Cohomology of the infinite-order jet space and the inverse problem. *J. Math. Phys.* **42**, 4272-4282 (2001)

[18] Giachetta, G., Mangiarotti, L. and Sardanashvily, G.: Lagrangian supersymmetries depending on derivatives. Global analysis and cohomology. *Commun. Math. Phys.* **259**, 103-128 (2005); *E-print arXiv*: hep-th/0407185

[19] Giachetta, G., Mangiarotti, L. and Sardanashvily, G.: *Geometric and Algebraic Topological Methods in Quantum Mechanics.* Singapore: World Scientific, 2005

[20] Gomis, J., París, J. and Samuel, S.: Antibracket, antifields and gauge theory quantization. *Phys. Rep* **295**, 1-145 (1995)

[21] Hernández Ruipérez, D. and Muñoz Masqué, J.: Global variational calculus on graded manifolds. *J. Math. Pures Appl.* **63**, 283-309 (1984)

[22] Hirzebruch, F.: *Topological Methods in Algebraic Geometry*. Berlin: Springer-Verlag, 1966

[23] Rennie, A.: Smoothness and locality for nonunital spectral triples. *K-Theory* **28**, 127-165 (2003)

[24] Sardanashvily, G.: Noether identities of a differential operator. The Koszul–Tate complex. *Int. J. Geom. Methods Mod. Phys.* **2**, N5 (2005); *E-print arXiv:* math.DG/0506103

[25] Takens, F.: A global version of the inverse problem of the calculus of variations. *J. Diff. Geom.* **14**, 543-562 (1979)